Part I

Appetizers (Before Calculus)

Chapter 1. Precalculus

Chapter 1

Precalculus

1.1 Using Parentheses with the Game of Telephone

Julie Barnes, Western Carolina University
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Concepts Taught: parentheses, order of operations, mathematical notation

Activity Overview

Most college students have been taught the order of operations and proper usage of notation, especially parentheses, well before coming to college. However, many students seem to have forgotten those rules and see no need for parentheses, writing out steps, or carrying through limits. This activity shows students the importance of correctly using proper mathematical notation. It is a lot like the childhood game of Telephone where the first child is given a message, whispers it to the next child, and after passing the message to 20-30 children in this fashion, the message is often distorted. Here, instead of whispering, each student works one step of the problem in isolation and passes the result to the next person, who in turn does the next step, and so on. If everyone uses proper notation and simplifies the mathematical expressions correctly, the "mathematical message" is transmitted without error. If parentheses are used incorrectly or any other notation errors are made, the answer that appears on the other end will most likely be incorrect. Students seem more willing to believe that the details of mathematical notation, such as parentheses, are necessary when their fellow students misinterpret their work rather than an instructor who may be perceived as being too picky.

Supplies Needed (per group)	Class Time Required	Group Size
Prepared problem* Prepared stack of sticky notes** roughly 1 sticky note per person	10 minutes	5-10 students depending on the problem

*This activity works best with each group doing the same problem at the same time. You could choose one from the four sample problems provided on page 7 with possible solutions on pages 8 and 9, but you are encouraged to choose a problem of your own that addresses any specific issues your students have encountered. Copy the problem you want your students to solve on several sticky notes, one sticky note per group.

**For each group, you will also need a stack of 5-10 sticky notes depending on the number of steps needed to simplify the problem chosen. Stick one of the prepared problems on the top of each stack. Add a few extra sticky notes to the bottom of the stack just in case your students do more steps than you had anticipated. It is helpful if each stack of sticky notes is a different color for future reference.

Running the Activity

Explain the game of Telephone and tell students that they will be doing one step toward solving a problem and presenting a final answer in simplified form. Hand one student per group a stack of sticky notes complete with a problem on the top sticky note. Tell that student to remove the top sticky note, stick it to his or her desk, and leave it there for the remainder of the activity. Then the student should write down the first step needed to evaluate or simplify the problem on the top sticky note remaining. After recording the first step, the student should pass the stack of sticky notes to another student in his or her group. The next student removes the top sticky note which has the previous students' work on it, sticks it to his or her desk, and leaves it there for the remainder of the activity. Then, he or she does the "next step" in simplifying the expression and writes it on the top sticky note of the remaining stack. Students continue this process until they feel the expression is simplified as far as it can be simplified.

Once everyone has finished, have students share their answers with the class. This activity is most helpful if some mistakes are made along the way and not all answers are the same. Assuming that not all answers are the same, ask students what happened and give them a chance to determine what mistakes were made. For the samples provided, common mistakes include leaving off the limit sign so that the answers still contain a variable, losing parentheses, and incorrectly substituting items into functions. Possible correct solutions to the sample problems are provided on pages 8 and 9; these could be projected via a document presenter in class to aid in the class discussion once students have determined what mistakes they think were made. In the rare event that all problems are solved perfectly, congratulate them for their accomplishment, and lead a discussion on why the mathematical ideas were transmitted without errors. In all cases, be sure that students walk away with the notion that details in mathematical notation are important.

Suggestions and Pitfalls

Make sure that students are not helping the student either before or after his or her step. That would eliminate the need for proper mathematical notation.

This activity also works well if pairs of students work on each step instead of individual students. That is, a pair of students would work on the first step, and pass their work to a different pair of students, and so on. Working in pairs is particularly helpful for students who are afraid of mathematics.

In lieu of a handout, you may want to ask your students to write a short paragraph about the importance of notation in mathematics.

Sample Problems

A. Simplify $-3(2-4)^2 + 24 \div 3 \times 2$.

B. Expand and simplify $-3x(x+1)(x-2)^2$.

C. Simplify $\dfrac{\frac{1}{x} - \frac{x+1}{x^2}}{x^3}$.

D. Simplify $\dfrac{\frac{1}{x-b} - \frac{1}{x}}{b}$.

E. Evaluate and simplify $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ where $f(x) = x^2 - 5$.

F. Let $f(x) = x^2 - 3x + x^3, g(x) = 1 - x$, and $h(x) = e^{-x}$.
Evaluate and simplify $f(-1)(g(2+c) + h(0))$.

G. Evaluate and simplify $\lim_{x\to 5} \dfrac{\frac{2}{x^2} - \frac{2}{25}}{x-5}$.

H. (*for calculus*) Evaluate and simplify $\dfrac{d}{dx}\left((x^2+x)^3\right)$.

Possible Correct Solutions to Sample Problems

A.

$$\begin{aligned} -3(2-4)^2 + 24 \div 3 \times 2 &= -3(-2)^2 + 24 \div 3 \times 2 \\ &= -3(4) + 24 \div 3 \times 2 \\ &= -3(4) + 8 \times 2 \\ &= -3(4) + 16 \\ &= -12 + 16 = 4. \end{aligned}$$

B.

$$\begin{aligned} &-3x(x+1)(x-2)^2 \\ &\quad = (-3x^2 - 3x)(x-2)^2 \\ &\quad = (-3x^2 - 3x)(x^2 - 4x + 4) \\ &\quad = -3x^4 + 12x^3 - 12x^2 - 3x^3 + 12x^2 - 12x \\ &\quad = -3x^4 + 9x^3 - 12x. \end{aligned}$$

C.

$$\begin{aligned} \frac{\frac{1}{x} - \frac{x+1}{x^2}}{x^3} &= \frac{\frac{x}{x^2} - \frac{x+1}{x^2}}{x^3} \\ &= \frac{\frac{x-(x+1)}{x^2}}{x^3} = \frac{\frac{x-x-1}{x^2}}{x^3} \\ &= \frac{\frac{-1}{x^2}}{x^3} = \frac{-1}{x^2} \cdot \frac{1}{x^3} \\ &= \frac{-1}{x^5}. \end{aligned}$$

D.

$$\begin{aligned} \frac{\frac{1}{x-b} - \frac{1}{x}}{b} &= \frac{\frac{x}{x(x-b)} - \frac{x-b}{x(x-b)}}{b} \\ &= \frac{\frac{x-(x-b)}{x(x-b)}}{b} = \frac{\frac{b}{x(x-b)}}{b} \\ &= \frac{b}{x(x-b)} \cdot \frac{1}{b} = \frac{1}{x(x-b)} \\ &= \frac{1}{x^2 - bx}. \end{aligned}$$

E. $$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{where } f(x) = x^2 - 5$$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{(x+h)^2 - 5 - (x^2-5)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 5 - (x^2-5)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 5 - x^2 + 5}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h\to 0} 2x + h = 2x.
\end{aligned}$$

F. $$\begin{aligned}
\lim_{x\to 5} \frac{\frac{2}{x^2} - \frac{2}{25}}{x-5} &= \lim_{x\to 5} \frac{\frac{50-2x^2}{25x^2}}{x-5} \\
&= \lim_{x\to 5} \frac{50-2x^2}{25x^2} \cdot \frac{1}{x-5} = \lim_{x\to 5} \frac{2(25-x^2)}{25x^2(x-5)} \\
&= \lim_{x\to 5} \frac{2(5-x)(5+x)}{25x^2(x-5)} = \lim_{x\to 5} \frac{-2(x-5)(5+x)}{25x^2(x-5)} \\
&= \lim_{x\to 5} \frac{-2(5+x)}{25x^2} = \lim_{x\to 5} \frac{-20}{625} = \frac{-4}{125}.
\end{aligned}$$

G. $$f(x) = x^2 - 3x + x^3, \quad g(x) = 1 - x, \quad h(x) = e^{-x}$$

$$\begin{aligned}
f(-1)(g(2+c) + h(0)) &= ((-1)^2 - 3(-1) + (-1)^3)(1 - (2+c) + e^{-0}) \\
&= (1 + 3 - 1)(1 - (2+c) + e^{-0}) \\
&= 3(1 - (2+c) + 1) \\
&= 3(2 - (2+c)) \\
&= 3(2 - 2 - c) \\
&= 3(-c) \\
&= -3c.
\end{aligned}$$

H. $$\begin{aligned}
\frac{d}{dx}\left((x^2+x)^3\right) &= 3(x^2+x)^2(2x+1) \\
&= 3(x^4 + 2x^3 + x^2)(2x+1) \\
&= (3x^4 + 6x^3 + 3x^2)(2x+1) \\
&= 6x^5 + 12x^4 + 6x^3 + 3x^4 + 6x^3 + 3x^2 \\
&= 6x^5 + 15x^4 + 12x^3 + 3x^2.
\end{aligned}$$

1.2 Function Ball Toss

Julie Barnes, Western Carolina University

Concepts Taught: functions, inverse functions, function tables

Activity Overview

Students in introductory college mathematics classes typically have seen graphs and equations of functions, but are not always comfortable with what it actually means to be a function. In this activity, students pass balls to each other as described by a table of data. In some cases, the rules for passing the balls represent a function and sometimes they do not. Students are able to physically see the difference between the two based on whether the rules are clear or not, giving them a concrete picture of what it means to be a function.

Supplies Needed	Class Time Required	Group Size
4 identical balls	10 minutes	Class demonstration

Running the Activity

In order to get students thinking about functions, have them imagine a candy machine. You put money in the machine and push a button. If the machine is working properly, exactly one item comes out; the machine is therefore a function. If two buttons yield the same type of candy, the machine is still a function because each button produces one result. However, if one button yields two types of candy, the machine is not working properly, and consequently, it is not a function.

Provide students with a copy of the handout to guide the discussion, and ask for four volunteers to stand in front of the room. Call the four people A, B, C, and D, and give each one a ball. Tell your class that the volunteers will be passing their balls according to specific rules provided from different tables. The job of each volunteer is to find his/her letter in the left column of a given table as well as the corresponding letter in the right column. On the count of three, each volunteer passes his/her ball to the person whose letter is in the right column. In Table 1, volunteers pass balls according to $f(x)$: A passes the ball to B, B passes the ball to C, C passes the ball to D, and D passes the ball to himself/herself. Figure 1.1 shows students acting out the description in Table 1. After the quick demonstration, give students a chance to complete Problem 1 from the handout and share their responses with others nearby. Then briefly discuss their answers as a class.

Once students realize that $f(x)$ from Table 1 is a function, have the volunteers reset such that each person has one ball, and then repeat the process with Table 2. Now, $g(x)$ is the rule that says A passes the ball to B, B passes the ball to C, C passes the ball to A, D passes the ball to A, and D passes the ball to D. When students are asked to pass their balls, D should notice quickly that he/she cannot do two things at once with the same ball. Have students complete Problem 2 and then share their responses with the class. They should notice that $g(x)$ is not a function because one element of the domain has unclear directions. Figure 1.2 shows students acting out the description in Table 2.

Figure 1.1: Students passing the ball according to Table 1 from the handout.

Figure 1.2: Students passing the ball according to Table 2 from the handout. Notice that Table 2 does not represent a function because Student D is not able to do what the table asks him to do.

Again have the volunteers reset so that each person has one ball. Have them act out Table 3 where the function $h(x)$ is the rule that says A passes the ball to B, B passes the ball to C, and C passes the ball to A. This leads to an interesting observation. D is standing there holding a ball and has no instructions. Have students complete Problem 3 and share answers with the class. Help them realize that this is not a function on all of the volunteers, $\{$A, B, C, D$\}$, but it is a function on the smaller domain of just $\{$A, B, C$\}$.

Finally, repeat the function ball toss described in Table 1. This time, have students respond to Problem 4 from the handout. Tie the notion of invertible functions to whether each person in the range knows who tossed the ball(s) to him/her. If a ball could have come from more than one source, then the function is not invertible.

Depending on the needs of your students, you could include more tables and/or have a longer discussion about what makes a process a function or not.

Suggestions and Pitfalls

Often, the hardest part is getting volunteers if your students are afraid of mathematics. It may be useful to emphasize that the volunteers only need to have basic ball tossing skills.

Note that we are using function notation in the tables even though some of the tables do not represent functions. If this is a concern, you may want to point this out to your students. Also, this activity could be modified to address other topics, such as determining if a function is one-to-one or onto.

Function Ball Toss – Class Handout

Table 1

x	$f(x)$
A	B
B	C
C	D
D	D

Table 2

x	$g(x)$
A	B
B	C
C	A
D	A
D	D

Table 3

x	$h(x)$
A	B
B	C
C	A

1. After the four volunteers demonstrate the action described by Table 1, complete the following problems.
   (a) A is not mentioned in the second column. What is unique about A's interaction with the ball toss? Does this keep $f(x)$ from being a function?
   (b) D is listed twice in the $f(x)$ column. What was different about D's interaction with the ball toss? Does this keep $f(x)$ from being a function?
   (c) Did all the volunteers know where they were supposed to toss the ball?
   (d) Does $f(x)$ represent a function? Why or why not?

2. After the four volunteers demonstrate the action described by Table 2, complete the following problems.
   (a) A is listed twice in the $g(x)$ column. What was unique about A's interaction with the ball toss?
   (b) D is listed twice in the x column. What was unique about D's interaction with the ball toss?
   (c) Did all the volunteers know where they were supposed to toss the ball?
   (d) Does $g(x)$ represent a function? Why or why not?

3. After the four volunteers demonstrate the action described by Table 3, complete the following problems.
   (a) Did all the volunteers know where they were supposed to toss the ball?
   (b) Does $h(x)$ represent a function on the domain $\{$A, B, C, D$\}$? Why or why not?
   (c) Is $h(x)$ a function on a different domain? Why or why not?

4. After the four volunteers repeat the procedure described in Table 1, complete the following problems.
   (a) Does B know who tossed the ball to him/her?
   (b) Are there any volunteers who received balls from multiple people?
   (c) If $f(x)$ is invertible, explain why; if not, present a modified version of $f(x)$ such that this new version is invertible.

1.3 Mathematical Modeling Using Long-Exposure Photography

Johann Thiel, New York City College of Technology

Concepts Taught: modeling, rate of change, linear functions, curve fitting

Activity Overview

An important and useful skill in multiple fields is the ability to construct appropriate mathematical models based on collected data. The goal of this activity is to create unique in-class data sets that students can then use to practice mathematical modeling. In particular, as this is an introductory lesson into the topic, we will focus on deciding whether or not a linear model is appropriate for a given data set. The instructor and students will use a digital camera and a flashing rubber ball to create data sets of the ball's position as a function of time. By setting the camera to a very slow shutter speed, the long-exposure images created will capture the motion of the flashing rubber ball in discrete steps.

Supplies Needed	Class Time Required	Group Size
Digital camera* Tripod Computer & projector Meter stick Flashing rubber ball**	15-20 minutes	2-3 students

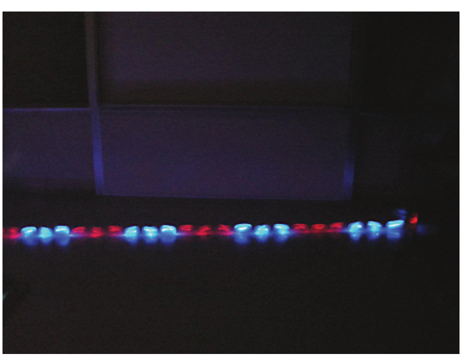
* The camera must have an adjustable shutter speed. Some smartphones are equipped with this feature.
** Flashing rubber ball toys can be purchased in pet stores, Walmart, Target, etc.
Note: We recommend that you test the camera and upload one or two shots onto the computer before class starts. This will help reduce the chances of technical issues arising in the middle of the activity. Also, before starting class, the flashing rubber ball should be photographed while in motion with the camera's shutter speed set to between 1-2 seconds. This is to help approximate the time interval between flashes, which can be computed as the shutter speed (in seconds) divided by the number of light points that appear in the photograph. For example, Figure 1.3b shows 23 flashes over an 8 second exposure, resulting in a period of approximately $\frac{1}{3}$s. The time period between flashes will be needed by the students in order to complete the handout.

Running the Activity

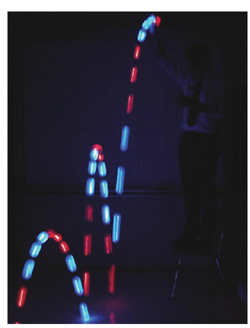
To begin the activity, set up the camera on the tripod facing a designated student holding the flashing rubber ball. Place the meter stick in a location that is as far away from the camera as the student. See Figure 1.3a for an example of the setup. Set the camera's shutter speed to at least four seconds, and dim the lights in the room.

For the first data set, photograph the movement of the flashing rubber ball as it rolls across the floor in front of the camera. For the second data set, photograph the movement of the flashing rubber ball as it is dropped from a height of at least five feet. The goal is to drop the ball with a slight horizontal velocity. This will make it easier to measure the ball's position as it falls for the first time. See Figures 1.3b and 1.3c for examples of resulting images.

(a) Example setup with meter stick.



(b) Flashing rubber ball rolling on the floor.



(c) Dropping the flashing rubber ball.

Figure 1.3: Long-exposure images of the flashing rubber ball.

Project the resulting images onto a screen that everyone can see. Have the class work together to measure and record the position of the ball relative to a fixed point for each flash. The class should now be able to start on the exercises in the handout. The handout guides students through the process of tabulating the data obtained from the images, graphing it, and answering questions on the use of appropriate mathematical models to describe the movement of the ball.

Once students are nearly finished with the handout, begin discussing some of the differences observed in the images. Students should be able to explain if they believe that a linear model suits either the rolled ball or the dropped ball. In the cases where a linear model appears to be a good fit, follow-up questions on how to construct the model should be pursued. In practice, this can lead to the introduction of derivatives and the acceleration due to gravity.

Suggestions and Pitfalls

For this lesson, when we discuss the position of the ball, we are referring to only one direction at a time. For example, in Figure 1.3b, position refers to the horizontal distance (x-position) of the ball over time, while in Figure 1.3c, position refers to the vertical height (y-position) of the ball over time. Depending on the case, when measuring the ball's position relative to a fixed point, we mean strictly either the horizontal or vertical position, but not both.

The position of the ball relative to a fixed point on the projected image can be measured using the meter stick by simply holding it up to the projected image. As it is unlikely that the projected image is to scale, the photographed meter stick from Figure 1.3a will help with converting the measured units to the appropriate scale.

If there is sufficient time, individual groups can make their own measurements. This can lead to interesting discussions about how to properly measure the position of the ball at any point in time. If such time is not available, you can still offer students the opportunity to take part in the process by having each come to the board to perform one position measurement that the entire class will use.

Mathematical Modeling Using Long-Exposure Photography – Class Handout

1. Fill out the tables below using the measurements collected in class. The columns are t for time, Δx and Δy for the changes in position over each time interval, and x and y for the positions relative to the starting point at time t.

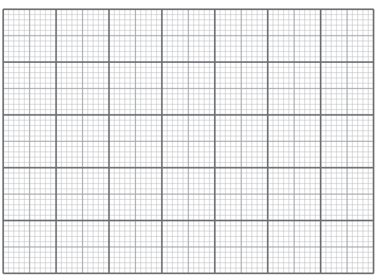
t	Δx	x

Table 1.1: Rolled ball.

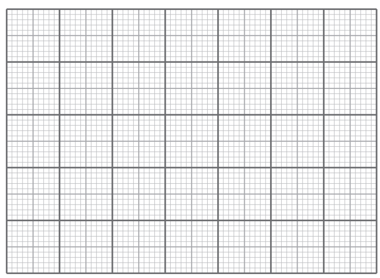
t	Δy	y

Table 1.2: Dropped ball.

2. Plot the position versus time (x vs. t) and (y vs. t) for each table on the space below. Be sure to label the axes appropriately.



x vs. t



y vs. t

3. Does either data set look like it can be modeled by a linear function $f(t) = mt + b$? If so, graphically estimate the value of m and b.

4. Compute the error between your model and the data. To do this, sum the absolute value of the difference between your model and the data at each recorded time. Can you change the value of m or b in your model to make the error smaller?

5. Repeat the above steps by plotting the changes in position versus time, Δx vs. t and Δy vs. t.

1.4 Function Composition Using Crackers and Cheese

Julie Barnes, Western Carolina University

Concepts Taught: function composition

Activity Overview

When students are introduced to function composition, it often has no meaning to them. Without any meaning, students may compose the wrong direction, turn composition into multiplication, or make any number of other mistakes. However, most students already have a very clear understanding of both the concept of a sandwich and how to make one. In this activity, sandwich making is broken down into two steps, each introduced as a function; students then use crackers and cheese to represent a variety of different compositions of the two functions. By changing the order of the sandwich-making steps, and relating this to function notation, students have an opportunity to make that connection between the notation for function composition and its meaning.

Supplies Needed (per group)	Class Time Required	Group Size
1 individual snack pack of crackers and cheese* Small paper plate	10 minutes	2-4 students

*The snack packs come in boxes and are convenient to use in class because they are easy to distribute and are the right size (roughly 5 or 6 crackers) for a group. It is also possible to use a large box of crackers and anything spreadable like cheese, jelly, or jam; for this to work, have a station where students can pick up the crackers, a plastic knife, and a spoonful of cheese, jelly, etc., on their plates for use during the activity.

Running the Activity

Provide each group with a paper plate, a snack pack of cheese and crackers, and a handout. Have students draw an "x" in the middle of the plate, as seen in Figure 1.5, and define the following two functions that will be used in this activity.

$C(x)$ is the function of placing a cracker on x.
$S(x)$ is the function of spreading one teaspoon of cheese on x.

It is helpful to have a brief discussion about the role of x as a placeholder and not just the mark on their plate. Ask questions such as, "What would C(chair) or C(hand) represent?"

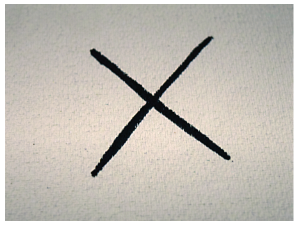
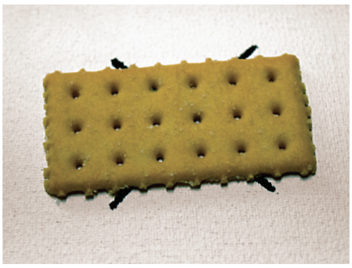
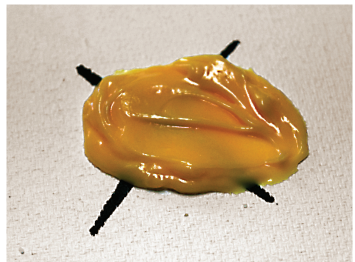


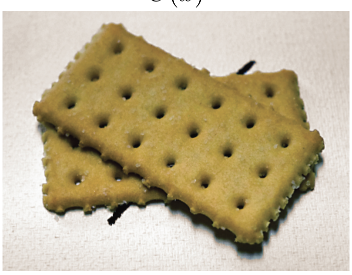
Figure 1.5: Starting by drawing an “X” on a paper plate.



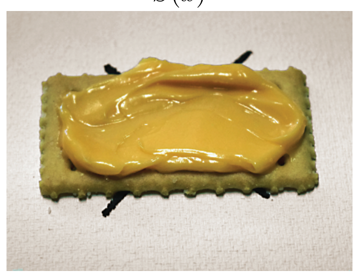
$C(x)$



$S(x)$

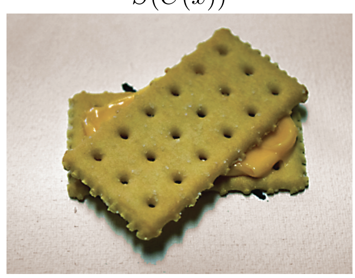


$C(C(x))$



$S(C(x))$

$S(C(C(x)))$



$C(S(C(x)))$

Figure 1.6: Photographs of constructions from the handout.

or “Would you want to create S(chair)? Why or why not?” Once everyone is comfortable with the notation, have them work through the problems on the handout. The handout asks students to explore a variety of possible compositions of $C(x)$ and $S(x)$ by creating each composition with their crackers and cheese. While students are working, circulate through the room to provide hints and answer questions. See Figure 1.6 for examples of the kinds of function compositions students will create.

Once the students have built their cracker creations, have a class discussion about the activity. What did they notice about the order of steps? How does this relate to the symbols describing what they did? How is composition different from basic addition? If they do not see a difference, ask them what $C(x) + S(x)$ would look like. It would have to involve two plates; one would have a cracker and one would have cheese spread on x. This is clearly different from what they just created.

Suggestions and Pitfalls

This is an easy activity to implement but it could generate crumbs; it is a good idea to bring paper towels or baby wipes. Also, be aware that students may have food allergies and some might not want to touch crackers or cheese for this reason. For a large class, you may want to do this as a class demonstration.

Function Composition using Crackers and Cheese – Class Handout

Before you begin, make sure that you have a paper plate and a snack pack of cheese and crackers. Draw an “x” in the center of your paper plate. We define the following two functions.

$C(x)$ is the function of placing a cracker on x.
$S(x)$ is the function of spreading one teaspoon of cheese on x.

1. Create $C(x)$. Describe what is on your plate.

2. Create $C(C(x))$. Describe what is on your plate. How does this differ from the previous problems?

3. Create $S(C(x))$. Describe what is on your plate.

4. Create $S(C(C(x)))$. Describe what is on your plate.

5. Make a cheese sandwich and place it on the “x” on your plate. What function symbols describe the sandwich you just created?

6. Would you want to create $C(S(x))$? Why or why not?

7. What is the difference between $C(S(x))$ and $S(C(x))$? Does order matter?

1.5 Walking Function Transformations

Julie Barnes, Western Carolina University
Kathy Jaqua, Western Carolina University

Concepts Taught: graphing function transformations, compositions of functions

Activity Overview

Once students have learned about functions and their related graphs, we often teach them how changes in the function translate into changes in the graph using the language of motion like *shifting*, *stretching*, *shrinking*, and *flipping*; yet we usually do not use the actual motions. In this activity, we give students an opportunity to become points on a Cartesian coordinate system and to walk through the motions dictated by transforming functions. In this way, students learn about shifting by shifting, stretching by stretching, and so on.

Supplies Needed	Class Time Required	Group Size
20 feet of adding machine paper* Tape Marker	50 minutes	3-4 students

*Painter's tape or string can be used instead of the adding machine paper.

Before class, rip the adding machine paper in half and tape it to the floor to represent an xy-coordinate system. Label units roughly one foot apart on the paper with a marker. If your classroom has moveable desks and a good amount of space, consider moving the desks to one side of the room to clear up some floor space.

Running the Activity

Provide each student with a copy of the handout and divide the class into groups of 3-4 students. Give each group an initial location with coordinates less than half of the largest x and y coordinates on your axes. Choose locations that are spread out on the coordinate system, making sure that one is on the x-axis and one is on the y-axis. Do Example 1 from Section 1 of the handout together. Then, have students complete Section 1 of the handout except for the final column. While they are working, walk around to answer questions and check their answers. If any groups finish early, have them help groups that are struggling.

Once all groups have completed Section 1, have one representative from each group stand at their given starting location and look at the first transformation of the handout. Have students *walk the transformation*, i.e., start at their given points and move to new locations as determined by the transformation. After walking the first transformation, have students describe what happened and fill in the last column on the chart in the handout. As you move on to the next transformation in Section 1, replace the volunteers so that by the end of the activity all students get a chance to participate; make sure the new walkers begin at their given starting points. See Figure 1.7 for photographs of where students would move under some of the transformations listed in the handout.

Once students have determined how to move under each type of transformation, challenge them to walk the transformations listed at the end of Section 1 without doing any computations.

Repeat this process for Section 2 of the handout.

At the end, ask students to explain or describe the difference between the transformations in Section 1 and Section 2 and challenge them to walk the transformations listed at the bottom which combine various transformations from both sections.

Suggestions and Pitfalls

Students tend to have more difficulty with transformations from Section 2 of the handout. For example, for $f(x-1)$, students often want to move to the left instead of the right. For $f(2x)$, they tend to want to stretch the function away from the y-axis instead of compressing it. Finally, a surprising thing may occur with $-f(x)$: students will sometimes stay in the same spot and spin 180 degrees. This is a good time to explain exactly what multiplying by a negative does.

This activity could also be done with more students walking if there is enough space. Just make room for them. For small classes, you can make the groups smaller or have students work individually.

For a more in-depth discussion, see [1] where this activity first appeared.

Reference

1. J. Barnes and K. Jaqua, Algebra aerobics, *Mathematics Teacher* 105 no. 2 (2011) 97–101.

$f(x)$

$2f(x)$

$-f(x)$

$f(x+1)$

$f(2x)$

$f(-x)$

Figure 1.7: Some examples of students walking transformations from the handout. Note that the positive x-axis is to the right and the students are facing the positive y-axis. The starting points for $f(x)$ are $(-4, 2)$, $(-2, 0)$, $(0, 1)$, $(1, -2)$, $(2, 2)$, and $(4, -1)$.

Walking Function Transformations – Class Handout

My group's initial location is $(x, f(x)) = ($________, ________$)$.

Section 1

In this section, fix your given x-coordinate and compute a new y-coordinate based on the transformation.

Example 1: Suppose the transformation is $f(x)+1$ and your starting location is $(5,6) = (x, f(x))$. Keep $x = 5$. Then, $y = f(x)+1 = f(5)+1 = 6+1 = 7$. The new location keeps $x = 5$, and now $y = 7$. Your new location is then $(5,7)$.

Determine where your location moves under each of the transformations below. Fill in the first three columns.

Transformation	Fixed x Coordinate x	New y Coordinate y	New Location (x, y)	Description of Change
$f(x)+1$				
$2f(x)$				
$\frac{1}{2}f(x)$				
$-f(x)$				

Stop here and wait for directions for whole class activity. You will be filling in the last column of the chart. Then some transformations you will be walking as a class are as follows.

$$f_1(x) = f(x)+3, \quad f_2(x) = f(x)-2, \quad f_3(x) = -2f(x), \quad f_4(x) = \frac{1}{2}f(x) - 2.$$

Section 2

In this section, fix your given y-coordinate and determine the new x-coordinate that would result in your fixed y coordinate after applying the function.

Suppose the transformation is $f(x+1)$ and your starting location is $(5,6) = (x, f(x))$. Keep $y = 6 = f(x+1) = f(5)$. Then, $x+1 = 5$, and $x = 4$. The new location keeps $y = 6$, and now $x = 4$. Your new location is then $(4,6)$.

Determine where your location moves under each of the transformations below. Fill in the first three columns.

Transformation	New x Coordinate x	Fixed y Coordinate y	New Location (x, y)	Description of Change
$f(x+1)$				
$f(2x)$				
$f(\frac{1}{2}x)$				
$f(-x)$				

Stop here and wait for directions for whole class activity. You will be filling in the last column of the chart. Then some transformations you will be walking as a class are as follows. $g_1 = f(x+3), g_2 = f(x-2), g_3 = f(-2x)$.

Final challenge: Without knowing your initial coordinates, how would you move when applying each of the following functions?

$$h_1(x) = f(x-1) + 2, \quad h_2(x) = 2f(x+1), \quad h_3(x) = f(2x) - 2.$$

1.6 Graphing Piecewise Functions with Feather Boas

Julie Barnes, Western Carolina University

Concepts Taught: piecewise functions, asymptotes, limits, graphs

Activity Overview

College students are fairly comfortable graphing standard linear and parabolic functions, if for no other reason than because they can punch some information into their calculator to obtain a good picture. However, the minute a function becomes piecewise, student confidence tends to dissipate.

In this activity, students work in small groups to graph two kinds of piecewise functions using feather boas. One of the functions is simply a combination of a linear piece and a quadratic piece. The other one is based on information provided about the limits of the function near asymptotes. It is easy to add more functions like these if you have time and your students could use more practice.

The beauty of using feather boas on the floor is that this type of group activity tends to catch the attention of students, making them want to participate. Since most students want to touch the feather boas, you typically do not have one student taking over and doing the activity alone. Also, the boa graphs are large enough that several students are able to work on them at the same time, and you are able to see how they are doing from a distance. In addition, the boas stay in place fairly well.

Supplies Needed (per group)	Class Time Required	Group Size
2 feather boas* 4 pieces of tape 10 feet of adding machine paper**	20-30 minutes	2-4 students

*Feather boas can be purchased at most craft stores. Alternatively, yarn or clothesline also work.
**Painter's tape can be used instead of the adding machine paper.

Running the Activity

At the beginning of class, provide each group with a handout, four pieces of tape, two feather boas, and ten feet of adding machine paper ripped into two five-feet sections. Have students find a place on the floor that is roughly 5 ft $\times$ 5 ft and tape the adding machine

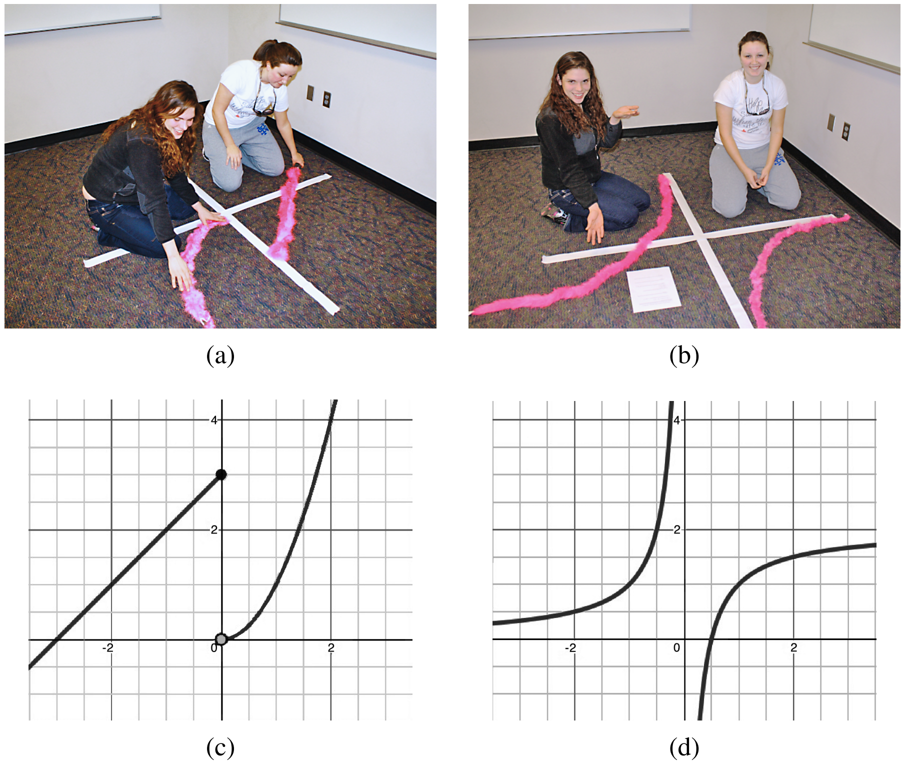


Figure 1.8: (a) Students working on each piece of the graph for $f(x)$. (b) Students displaying their graph for $g(x)$. (c) The graph for $f(x)$. (d) A possible graph for $g(x)$. Note that the graphs in (a) and (b) are oriented toward the students.

paper to the floor in such a way that it looks like a set of xy-axes as seen in Figures 1.8a and 1.8b.

Once their axes are ready, students should graph the problems stated on the handout. As they work, walk around the room to ask students questions about their graphs and have them explain their thinking. Be sure to have them indicate which is the x-axis and which is the y-axis, so everyone is reading it from the same direction. Examples of student work are shown in Figure 1.8, with Figures 1.8c and 1.8d included to show the graphs from the perspective of the students.

The goal here is to help students think through the meaning of a piecewise function and its corresponding graph; it is not just for students to generate perfect graphs, since that could be done more accurately with pencil and paper. In addition, since the boas are movable, students are able to try different ideas until they get the graph correct without needing to use an eraser. Also, capitalize on the fact that there are separate boas for each piece of the graph, as this allows students to think about the graph one piece at a time.

Suggestions and Pitfalls

When graphing $f(x)$, it is important to explain that the boas do not capture the open and closed circles (shown in Figure 1.8c). If desired, these features can be captured using additional props, such as canning rings for open circles and canning lids for closed circles.

Space is sometimes an issue. If your classroom is large enough with movable desks, have the students move the desks aside to work on the floor. Depending on logistics at your institution, you may be able to move into the hallway or any other nearby spaces like conference rooms or study areas. Alternatively, if the room has large tables, students can work on the tables instead of the floor.

Students often have a lot of questions. Therefore if you have a large class, it is best if you enlist a colleague or teaching assistant to help you on the day of the activity, as this is a great, non-threatening opportunity to provide some individual assistance while students work with the feather boas.

This activity can also be done in groups at their desks using Wikki Stix, Bendaroos, pipe cleaners, or even yarn. Wikki Stix and Bendaroos are available online and in some craft stores. These bendable sticks have the added feature that they adhere to the page but can also be removed much like sticky notes.

Graphing Piecewise Functions with Feather Boas – Class Handout

Use two pieces of adding machine paper that are each about five feet long in order to create an xy-coordinate system on the floor. Tape it in place. Then use two feather boas to create each of the graphs described below. When you finish creating each graph, sketch the graph and answer the questions.

1. Use feather boas to graph $f(x) = \begin{cases} x+3 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0. \end{cases}$

   Sketch a copy of your graph.

   (a) How does this graph relate to the individual graphs for $y = x + 3$ and $y = x^2$?

   (b) Why were you given two feather boas? Could this function have been graphed with only one boa? Why or why not?

   (c) Is your graph a function? How can you tell?

2. Use feather boas to graph a function that satisfies all of the following properties:

   $\lim_{x\to\infty} g(x) = 2$
   $\lim_{x\to 0^+} g(x) = -\infty$
   $\lim_{x\to 0^-} g(x) = \infty$
   $\lim_{x\to -\infty} g(x) = 0$

   Sketch a copy of your graph.

   (a) Are there any vertical asymptotes? If so, what are their equation(s)? How are they related to the limits?

   (b) Are there any horizontal asymptotes? If so, what are their equation(s)? How are they related to the limits?

   (c) Why were you given two feather boas? Could the function have been graphed with only one boa? Why or why not?

   (d) Is your graph a function? How can you tell?

1.7 Trigonometry Parameter Comparisons

Julie Barnes, Western Carolina University
Jessica Libertini, Virginia Military Institute

Concepts Taught: sine functions, amplitude, period, frequency, vertical shifts, phase shifts

Activity Overview

Students are typically able to graph trigonometric functions on a graphing calculator, and they can use the formulas for computing amplitude, period, vertical shifts, phase shifts, and frequency. However, they tend to have more difficulty explaining what the parameters of the functions mean and how they affect the corresponding graphs. In this activity, each student takes ownership of a particular sinusoidal graph and develops equation(s) that could generate his/her graph. In the process, students analyze the effect of the different parameters found in the the standard equation $y = a\sin(bx + c) + d$.

Supplies Needed	Class Time Required	Group Size
Prepared graph cards*	10-15 minutes	1-2 students

*Before the activity, photocopy the graphs from the table provided on pages 34 and 35 onto cardstock and use scissors to cut the graphs. You need one graph per group.

Running the Activity

This activity assumes that students have been introduced to the concepts and terminology of amplitude, period, vertical shifts, phase shifts, and frequency, as well as standard form for equations of sine graphs, but that they have had limited experience with these concepts.

Start by providing each group with one of the prepared graph cards and a handout. The handout then directs students as they work through four steps to create equations for their graphs. In the first step, students determine the amplitude, find other students whose graphs have the same amplitude, and then answer the questions from their handout concerning amplitude. Once everyone has completed this, students move to the second step where they determine the period of their graphs, find other students whose graphs have the same period, and answer the questions concerning the period of their graphs. Have students continue in this fashion as the handout guides them through a discussion about vertical shifts and phase shifts.

While students work, circulate around the room to help students find their groups and provide hints as needed. You may want to have some whole-class discussions to distill the main points after each step. Note that if you use the graphs provided, there should be four groups of four graphs in each of the first three steps. However, since phase shifts are not unique, there might be different sized groups of graphs in the last step. Students tend to have the most difficulty in determining other ways of computing phase shifts, so be prepared to provide hints like, “What would happen if you thought of it as a shift to the left?” or “What would happen if you add the period to your phase shift?”

When everyone has finished writing equations for their graphs, display them around the room, either by placing them on different desks, propping them on a chalkboard sill, or even taping them to the walls. Students can then walk around and compare the different equations with the graphs.

Suggestions and Pitfalls

There are 16 different graphs provided at the end of this section. If you have a smaller class, feel free to choose a subset of these graphs. For larger classes, students can work in groups or you can create more graphs for them to use.

If you want to do a similar activity for a wider collection of trigonometric functions like cosine curves or the negative sine function, feel free to add more graphs. If you do that, you may want to drop the phase shift portion of the activity, or simply remind your students that there is more than one trigonometric function that can be used to describe each graph.

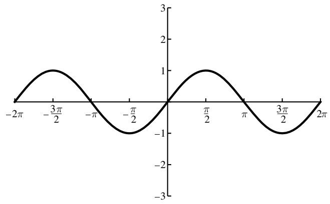
On the top of the handout provided, the standard form for the sine function is written as $y = a\sin(bx + c) + d$. If you use a different standard form, you may want to make changes accordingly.

The equations used to generate the graphs on pages 34-35 are listed in Table 1.3. Note that by changing phase shifts, there is more than one possible equation for each of the graphs.

Table 1.3: Equations used to generate graphs found on pages 34 and 35.

A. $y = \sin(x)$	B. $y = \sin(2x - \pi) + 1$	C. $y = 2\sin(4x)$
D. $y = 2\sin\left(x - \frac{\pi}{2}\right) + 1$	E. $y = \frac{1}{2}\sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$	F. $y = \frac{1}{2}\sin\left(4x - \pi\right) + 1$
G. $y = \frac{3}{2}\sin(2x - \pi)$	H. $y = \frac{3}{2}\sin\left(\frac{1}{2}x\right) + 1$	I. $y = \sin\left(\frac{1}{2}x + \frac{\pi}{4}\right) + \frac{1}{2}$
J. $y = \sin(4x) - 1$	K. $y = 2\sin(2x + \pi) + \frac{1}{2}$	L. $y = 2\sin\left(\frac{1}{2}x\right) - 1$
M. $y = \frac{1}{2}\sin\left(x + \frac{\pi}{2}\right) + \frac{1}{2}$	N. $y = \frac{1}{2}\sin\left(2x - \frac{\pi}{2}\right) - 1$	O. $y = \frac{3}{2}\sin(4x - \pi) + \frac{1}{2}$
P. $y = \frac{3}{2}\sin\left(x + \frac{\pi}{2}\right) - 1$		

A.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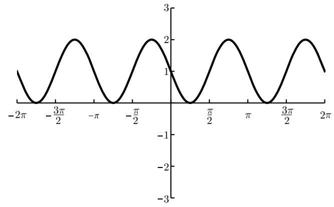
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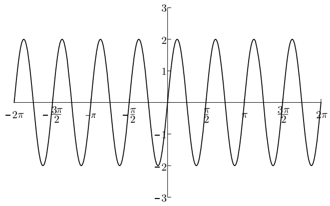
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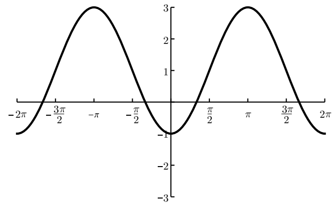
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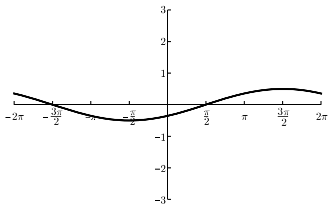
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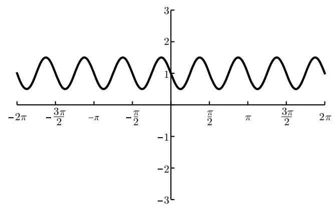
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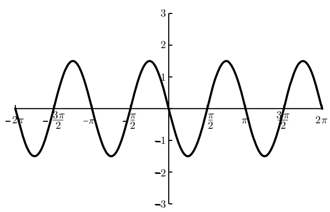
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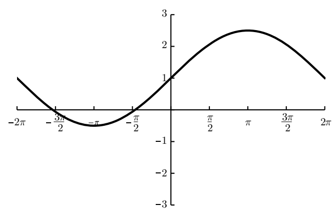
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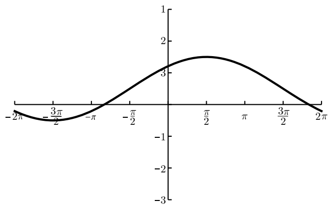
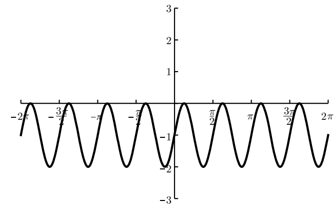
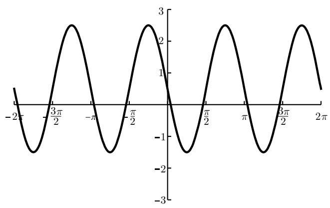
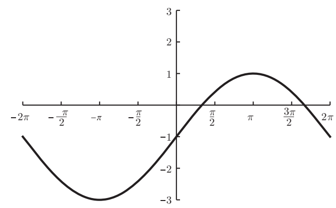
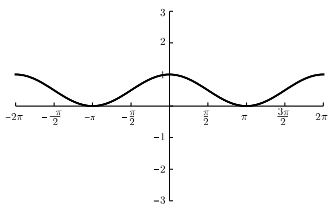
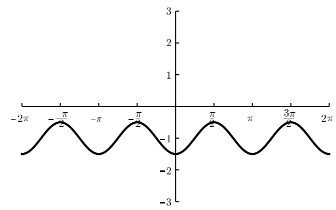
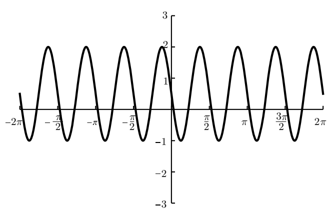
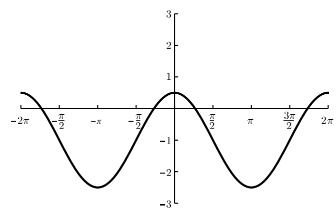
I. 	J. 
K. 	L. 
M. 	N. 
O. 	P. 

Trigonometry Parameter Comparisons – Class Handout

In this activity, you will go through several steps to develop an equation for your graph. All functions shown could be written in the form $y = a\sin(bx + c) + d$.

Amplitude

1. Determine the amplitude of your graph. Then, find all of your classmates who have graphs of functions with the same amplitude.

2. Describe the similarities in the collection of graphs that have the same amplitude as your graph.

3. For each graph in your group, determine the maximum value (height), the minimum value, and the value of the maximum value minus the minimum value. How is this calculation related to the amplitude?

4. Use the amplitude to determine the a portion of your equation, and then wait for your instructor to ask you to move to the next step.

Period

1. Determine the period of your graph. Then, find all of your classmates who have graphs of functions with the same period.

2. Describe the similarities in the collection of graphs that have the same period as your graph.

3. For each graph in your group, calculate the frequency. How does this number compare to the period? What is the difference between period and frequency graphically?

4. Use the period to determine the b portion of your equation, and then wait for your instructor to ask you to move to the next step.

Vertical Shift

1. Determine the vertical shift of your graph, keeping in mind that there might not be a vertical shift (i.e., the vertical shift is zero). Then, find all of your classmates who have graphs of functions with the same vertical shift.

2. Describe the similarities in the collection of graphs in your group that have the same vertical shift.

3. Use the vertical shift to determine the d value in your equation, and then wait for your instructor to ask you to move to the next step.

Phase Shifts

1. Determine a phase shift for your graph, keeping in mind that there might not be a phase shift (i.e., the phase shift is 0). Then, find all of your classmates who have graphs of functions with the same phase shift. If you are having trouble finding anyone, try computing another value for the phase shift.

2. Describe the similarities in the collection of graphs in your group that have the same phase shifts.

3. As a group, determine two other phase shifts that would generate the same graph. How could you describe an infinite number of other phase shifts that could also be used to generate the same graph?

4. Use one of your phase shifts to determine a value for c in your equation.