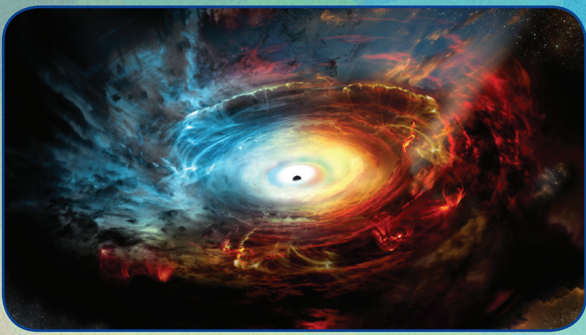
American Mathematical Society

CURRENT EVENTS BULLETIN

Friday, January 6, 2017, 1:00 PM to 5:00 PM

Imperial Ballroom A, Marquis Level, Marriott Marquis Atlanta

Joint Mathematics Meetings, Atlanta, GA

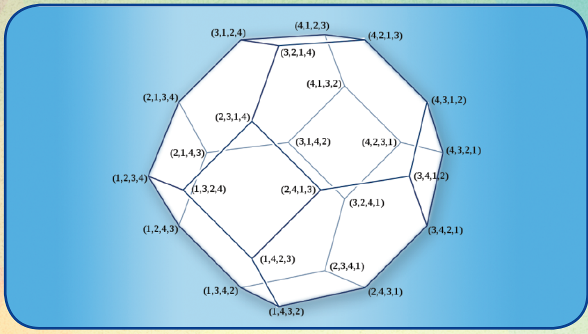


1:00 PM

Lydia Bieri, *University of Michigan*

Black hole formation and stability: a mathematical investigation.

Mathematics seems to be the only way to shed light on the weirdness of the universe.

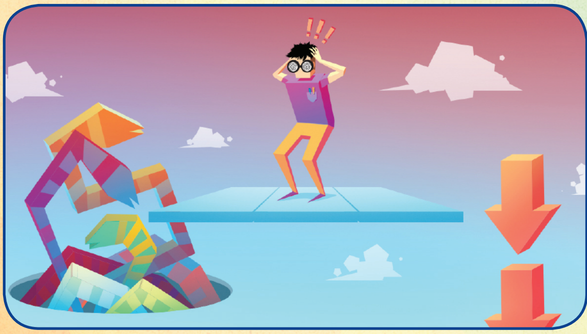


2:00 PM

Matt Baker, *Georgia Tech*

Hodge Theory in Combinatorics.

Ideas from complex analytic geometry solve a beautiful old problem in combinatorics.



3:00 PM

Kannan Soundararajan, *Stanford University*

Tao's resolution of the Erdős Discrepancy Problem.

Analysis and number theory to the rescue to understand ± 1 sequences.

4:00 PM

Susan Holmes, *Stanford University*

Statistical proof? The problem of irreproducibility.

Why medical (and other scientific) "results" seem to degrade with time—and what to do about it.

Organized by David Eisenbud, University of California, Berkeley

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Introduction to the Current Events Bulletin

Will the Riemann Hypothesis be proved this week? What is the Geometric Langlands Conjecture about? How could you best exploit a stream of data flowing by too fast to capture? I think we mathematicians are provoked to ask such questions by our sense that underneath the vastness of mathematics is a fundamental unity allowing us to look into many different corners -- though we couldn't possibly work in all of them. I love the idea of having an expert explain such things to me in a brief, accessible way. And I, like most of us, love common-room gossip.

The Current Events Bulletin Session at the Joint Mathematics Meetings, begun in 2003, is an event where the speakers do not report on their own work, but survey some of the most interesting current developments in mathematics, pure and applied. The wonderful tradition of the Bourbaki Seminar is an inspiration, but we aim for more accessible treatments and a wider range of subjects. I've been the organizer of these sessions since they started, but a varying, broadly constituted advisory committee helps select the topics and speakers. Excellence in exposition is a prime consideration.

A written exposition greatly increases the number of people who can enjoy the product of the sessions, so speakers are asked to do the hard work of producing such articles. These are made into a booklet distributed at the meeting. Speakers are then invited to submit papers based on them to the *Bulletin of the AMS*, and this has led to many fine publications.

I hope you'll enjoy the papers produced from these sessions, but there's nothing like being at the talks -- don't miss them!

David Eisenbud, Organizer
Mathematical Sciences Research Institute
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For PDF files of talks given in prior years, see
http://www.ams.org/ams/current-events-bulletin.html.
The list of speakers/titles from prior years may be found at the end of this booklet.

Black hole formation and stability: a mathematical investigation

Lydia Bieri

Abstract: The dynamics of the Einstein equations feature the formation of black holes. The latter are related to the presence of trapped surfaces in the spacetime manifold. The mathematical study of these phenomena has gained momentum since D. Christodoulou's breakthrough result proving that in the regime of pure general relativity trapped surfaces form through the focusing of gravitational waves. (The latter were observed for the first time in 2015 by LIGO.) The proof combines new ideas from geometric analysis and nonlinear partial differential equations as well as it introduces new methods to solve large data problems. These methods have many applications beyond general relativity. D. Christodoulou's result was generalized by S. Klainerman and I. Rodnianski. Here, we investigate the dynamics of the Einstein equations, focusing on these works. Moreover, we address the question of stability of black holes and what has been known so far, involving recent works of many contributors.

Contents

1 Introduction

The Einstein equations exhibit singularities that are hidden behind event horizons of black holes. A black hole is a region of spacetime that cannot be observed from infinity. The first encounters with its intriguing properties go back to the famous Schwarzschild solution of the Einstein vacuum (EV) equations in four spacetime dimensions

$$Ric(g) = 0\ . \tag{1}$$

In 1916, K. Schwarzschild [78] found the first solution to these equations, after A. Einstein in 1915 had formulated the general theory of relativity and had derived the Einstein equations [33], [34]. The Schwarzschild solution is spherically symmetric and depends on the mass M. Very important, Birkhoff [12] in 1923 proved that the Schwarzschild solution is the only spherically symmetric solution of the EV equations. It describes the gravitational field outside a non-rotating star or black hole, generally outside of any spherically symmetric body. The evolution of the body itself does not change the gravitational field in the exterior. We note at this point that a spherically symmetric object does not generate any gravitational waves. In the coordinate system, in which the Schwarzschild solution was first discovered, it has a singularity at $r = 2M$, where r denotes the radius of the spheres being the orbits of the rotation group. In 1924, A. Eddington [32] used a coordinate transformation getting rid of this singularity but did not comment on it. It was G. Lemaître [58] who in 1933 observed that this is only a coordinate singularity and that the Schwarzschild solution behaves "nicely" there in other coordinates. There is a true singularity at $r = 0$. By works of D. Finkelstein [40], M.D. Kruskal [55], J.L. Synge [79], G. Szekeres [84] systems of coordinates for the complete analytic extension of the Schwarzschild solution had been found, and dynamical properties of the region $r < 2M$ had been addressed. Finkelstein [40] in 1958 mentioned that the hypersurface $r = 2M$ is an event horizon, that is the boundary of the region, which is causally connected to infinity. More interesting is the behavior of other black hole spacetimes, namely the Kerr solutions discovered by R. Kerr [47] in 1963. This is a two-parameter-family of axi-symmetric solutions of the EV equations (1) having an event horizon, and the spacetime outside this horizon is a regular asymptotically flat region. Besides the mass parameter M (positive) this family is characterized also by the angular momentum a about the axis of symmetry with $|a| \leq M^2$.

Are black holes rare phenomena or do we expect them to occur often in the universe? How do they form and can they form in the evolution of initial data that do not contain any black holes? The latter was investigated by Demetrios Christodoulou in 2008 in his celebrated monograph [24]. Christodoulou's answer is yes and he provides a detailed description of black hole formation in his main proof. In order to state the main result of [24], we now turn to the notion of a *closed trapped surface.*

A concept directly related to the formation of black holes is a closed trapped surface introduced by R. Penrose [67] in 1965. He defines a trapped surface to be a spacelike surface such that the expansion scalars with respect to every family of future-directed null geodesic normals are negative, i.e. infinitesimally virtual displacements along these normals imply pointwise decrease of the area element. Penrose proved that

Theorem 1 (R. Penrose [67]) *A spacetime (M, g) is future null geodesically incomplete if the following three conditions hold:*

1. *$Ric(V, V) \geq 0$ for all null vectors V.*
2. *There exists a non-compact Cauchy hypersurface H in M.*
3. *There is a closed trapped surface S in M.*

A modern version of this incompleteness theorem can be formulated as follows:

Theorem 2 *Consider regular characteristic initial data on a complete null geodesic cone $\mathcal{C}$. Denote by (M, g) the maximal future development of the data on $\mathcal{C}$. Assume that M contains a closed trapped surface S. Then (M, g) is future null geodesically incomplete.*

At this point it was not clear at all if closed trapped surfaces form in the evolution of data that does not contain any such surface. In particular, one can ask what happens in the situation where the initial conditions are very far from containing a closed trapped surface. We would like to study the long-time evolution for the Einstein equations and show that closed trapped surfaces form under physical conditions. Christodoulou did this through analyzing the dynamics of gravitational collapse.

The first and simplest version of Christodoulou's main result on the formation of closed trapped surfaces for the Einstein vacuum equations (1) can be stated as follows:

Theorem 3 (D. Christodoulou [24]) *Closed trapped surfaces form in the Cauchy development of initial data, which are arbitrarily dispersed, if the incoming energy per unit solid angle in each direction in a suitably small time interval is large enough.*

We can rephrase this result, saying that enough energy through gravitational waves has to be concentrated in a small enough region, then a closed trapped surface will form.

Christodoulou's result was generalized by Sergiu Klainerman and Igor Rodnianski [51] to allow for more general initial data.

Next, we can ask if black holes are stable. This question is topic of ongoing research in the field. Many contributors have studied the first step towards understanding this problem.

In the present notes, we will first give an introduction to the main ideas of mathematical general relativity, then investigate the main steps and methods of the proof of black hole formation by Christodoulou and the generalization by Klainerman and Rodnianski, then we shall address the stability problem of black holes. Along the way, we will highlight gravitational waves that were observed for the first time in 2015 by LIGO [1].

2 Mathematical General Relativity

2.1 Einstein Equations and Spacetime Manifold

The Equations. Albert Einstein in 1915 derived the famous field equations of gravitation and established the general relativity (GR) theory [33], [34]. Much different from Newtonian physics, where space and time are separate and independent concepts, already special relativity (1905) combines space and time into a (flat) spacetime manifold known as the Minkowski spacetime. General relativity gives that manifold a curved metric whose curvature encodes the properties of the gravitational field. Thus gravitation acts through curvature. Our spacetime manifold carries all the information. "It is all there is, and nothing lives independently from it." The Newtonian law under which each mass makes a gravitational field is replaced by the Einstein field equations

$$R_{ij} - \tfrac{1}{2} R g_{ij} = \frac{8\pi G}{c^4} T_{ij} \ . \tag{2}$$

Here c denotes the speed of light, G is the Newtonian gravitational constant, the indices i, j take on values $0, 1, 2, 3$ and the tensors are as follows: R_{ij} is the Ricci curvature tensor, R the scalar curvature tensor, g_{ij} the metric tensor and T_{ij} denotes the energy-momentum tensor. The latter contains matter or energy present in the spacetime such as a fluid or electromagnetic fields. If there are matter (or energy) fields, thus $T_{ij} \neq 0$, then they obey their own evolution equations and together with the Einstein equations (2) form a coupled system. One then solves the Einstein system for the metric tensor g_{ij}. If there are no other fields, then $T_{ij} = 0$ and (2) reduce to the Einstein vacuum (EV) equations (1).

The Spacetime. A crucial difference from studying a pde on a (Euclidean or curved) background is that we are constructing the manifold itself by solving the Einstein equations. The resulting spacetime may feature intriguing properties including black hole formation or gravitational waves. The main goal of mathematical GR is to investigate classes of these manifolds, their structures and dynamics as well as their stability. This can only be achieved by solving the Cauchy problem for physical settings via geometric analysis and often combining various areas of mathematics.

Definition 1 *A spacetime manifold is defined to be a* 4*-dimensional, oriented, differentiable manifold* M *with a Lorentzian metric* g*.*

Remark: An n-dimensional spacetime is defined in the corresponding way.

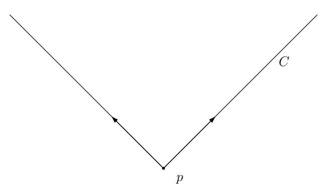
Definition 2 *A Lorentzian metric* g *is defined to be a continuous assignment of a non-degenerate quadratic form* g_p*, being of index one, in* T_pM *for every* p *in* M*.*

The simplest example of a Lorentzian metric is the Minkowski metric $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$. In our Lorentzian manifold (M, g) the tangent space T_pM at each p is flat Minkwoskian. We observe that a Lorentzian metric is a special case of a pseudo-Riemannian metric.

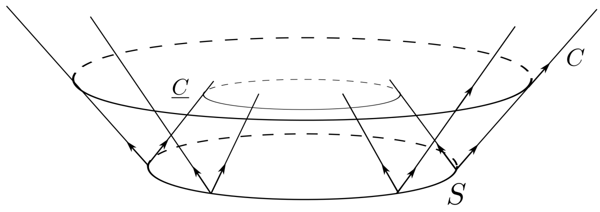
In these notes, we are interested in purely gravitational questions. Therefore, we are looking for spacetimes (M, g) with g solving the EV equations (1). At this point, we note that these equations can be written as a system of nonlinear, hyperbolic partial differential equations (pde).

The Lorentzian structure of our metric allows us to distinguish the following three types of vectors at a point p in T_pM: A vector $X \in T_pM$ is called timelike if $g_p(X, X) < 0$, it is called null or lightlike if $g_p(X, X) = 0$, and spacelike if $g_p(X, X) > 0$. A hypersurface is called spacelike if its normal vector is timelike. As in GR nothing travels faster than the speed of light, trajectories of massless particles are null whereas those for massive objects are timelike. A curve that is timelike or null is called *causal*.

We know that in special relativity light travels along light cones.



In general relativity light travels along null hypersurfaces which are generated by the congruence of null geodesics.

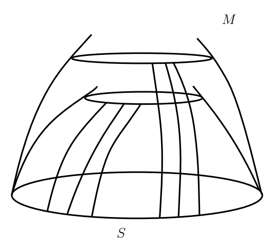


Let us start with a spacelike 2-surface S in M. At each $p \in T_pM$ we identify two orthogonal future-directed null vectors, namely $\mathcal{L}_p$ which is outward pointing, and $\underline{\mathcal{L}}_p$ which is inward pointing. The corresponding vectorfields defined in this way on S are $\mathcal{L}$ respectively $\underline{\mathcal{L}}$. The null hypersurfaces generated by the corresponding sets of null geodesics orthogonal to S are denoted by C and $\underline{C}$. Viewing S as a hypersurface in C, we denote its second fundamental form by χ, and the second fundamental form of S in $\underline{C}$ by $\underline{\chi}$. Their traceless parts are called the shears and denoted by $\hat{\chi}$, $\underline{\hat{\chi}}$ respectively. The traces $tr\chi$ and $tr\underline{\chi}$ are the expansion scalars.

From above and with this notation we know that S is trapped if

$$tr\chi < 0 \quad and \quad tr\underline{\chi} < 0 \quad .$$

Thus the null hypersurfaces look as in the following figure.



The mathematical structures behind this picture will be explained in section 4.2. For the moment, let us note that in order to prove theorem 3, Christodoulou in [24] uses a "natural" foliation of the spacetime into null hypersurfaces C_u and $\underline{C}_{\underline{u}}$, the double null foliation.

We can also think of S as the intersection of C_u with a spacelike hypersurface H_t, where the C_u are the null hypersurfaces of a foliation of the spacetime (M, g) with respect to an optical function u. Then we write $S_{t,u}$ for the 2-surface. Note that the $S_{t,u}$ are diffeomorphic to S^2. The main foliations of the spacetime in [27] are given first by the time function t, of which the level sets are maximal spacelike hypersurfaces H_t with vanishing linear momentum, second by the optical function u (known as retarded time), for which the level sets are the C_u. The foliations are such that the density of the $S_{t,u}$ in the $\mathcal{H}_t$ tends to 1 as $t \to \infty$.

These foliations were crucial in the works by Christodoulou and Klainerman [27] on the global nonlinear stability of Minkowski space. Christodoulou in [24] combines the double-null structure and methods from [27] with new features to investigate black hole formation. Beyond that, these structures prove natural to describe radiation. In a radiative spacetime, gravitational waves (fluctuations of the curvature) travel along these null hypersurfaces C_u from their sources such as mergers of binary black holes, of neutron star or as core-collapse supernovae. When we observe these waves, we can think of ourselves as located at *future null infinity* $\mathcal{I}^+$ which is defined to be the endpoints of all future directed null geodesics along which for the surfaces S the area radius $r \to \infty$. It has the topology of $\mathbb{R} \times \mathbb{S}^2$ with the function u taking values in $\mathbb{R}$. In other words, the null hypersurface C_u intersects $\mathcal{I}^+$ at infinity in a 2-sphere $S_{\infty,u}$.

Let us come back to the notion of black holes again. In general, we define the black hole region of an asymptotically flat spacetime (M, g) to be the set of points $B \subset M$ not in the past of *future null infinity* $\mathcal{I}^+$. We write $B = M \backslash J^-(\mathcal{I}^+)$. [1]

Above we encountered the Schwarzschild and Kerr black holes. In these cases, all causal geodesics $c(s)$ entering B are incomplete towards the future. We say that (M, g) is future-causally geodesically incomplete. In the Schwarzschild solution the curvature grows along all incomplete $c(s)$ when the affine parameter s tends to its supremum. The situation for Kerr is more colorful, as causality breaks down.

2.2 Cauchy Problem

The above mentioned Schwarzschild and Kerr spacetimes are examples of exact solutions of the Einstein equations. Whereas there exist quite a few closed-form solutions, many important physical situations do not have them. Nor do they help to investigate the space of solutions nor understand the dynamics of GR. In order to study stability problems, gravitational waves and questions about the dynamics of the gravitational field, we have to solve the Cauchy problem. Exact solutions certainly inspired insights into particular cases, but only solving the Cauchy problem can answer these important questions. We shall see that for the Einstein equations this mainly means proving theorems with geometric-analytic methods. Whereas in other fields of pde analysis, geometry plays less of a role, the geometric nature of GR is crucial and features extra challenges, but it also can be used to our advantage to obtain estimates.

Within GR, other techniques have been used to approximate solutions to the Einstein equations. Among them we find methods in perturbation theory and numerical relativity. Even though the main focus in these other fields is not on proving theorems, they provide important insights into physical problems.

It is clear that all these endeavors have to be understood within the larger realm of the Cauchy problem for the Einstein equations. Moreover, only the mathematical treatment of the latter yields a full understanding of the physical picture with its intricate features.

Let us consider a solution of the EV equations (1). Denote by ∇ the covariant derivative. Then the Bianchi identities

$$\nabla_{[\alpha} R_{\beta\gamma]\delta\epsilon} := \nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0 \tag{3}$$

are equivalent to the contracted Bianchi identities

$$\nabla^\alpha R_{\alpha\beta\gamma\delta} = 0 \tag{4}$$

with $\nabla^\alpha := (g^{-1})^{\alpha\beta} \nabla_\beta$.

Under the 4 constraints from the Bianchi identities, the EV system (1) provide 6 independent equations for the 10 unknowns of the metric g_{ij}. Here, we encounter the general covariance of the Einstein equations and remark that uniqueness of solutions holds up to

[1] This notion of a black hole actually "has to be investigated more". We address some issues in this article. For a nice discussion of black hole spacetimes see [29].

the equivalence under diffeomorphisms. This mathematical fact has the following physical meaning: The laws of nature do not depend on the coordinates.

Classic Approach. The Einstein equations split into a set of constraint equations that the initial data have to obey and a set of evolution equations. The "classic" initial value problem in GR considers a 3-dimensional manifold with a complete Riemannian metric $\bar{g}$ and a symmetric 2-tensor k solving the constraint equations

$$\bar{\nabla}^i k_{ij} - \bar{\nabla}_j trk = 0 \tag{5}$$
$$\bar{R} + (trk)^2 - |k|^2 = 0 , \tag{6}$$

where barred quantities are with respect to H. The data evolves according to

$$\frac{\partial \bar{g}_{ij}}{\partial t} = -2\Phi k_{ij} + \mathcal{L}_X \bar{g}_{ij} \tag{7}$$
$$\frac{\partial k_{ij}}{\partial t} = (\bar{R}_{ij} + k_{ij} trk - 2k_{is}k^s_j)\Phi + \mathcal{L}_X k_{ij} - \bar{\nabla}_i \bar{\nabla}_j \Phi \tag{8}$$

with $\Phi := 1/\sqrt{-g^{ij}\partial_i t \partial_j t}$ denoting the lapse function and X the shift vector. The time vector field is $T = \Phi N + X$ and $\mathcal{L}$ is the Lie derivative. The initial data set $(H, \bar{g}_{ij}, k_{ij})$ embeds into the Cauchy development (M, g), namely the Lorentzian spacetime, as a spacelike hypersurface. The imbedding $H \to M$ has first respectively second fundamental forms $i_*(\bar{g})$ and $i_*(k)$. From this point of view, it is easy to see that the constraint equations (5) - (6) then are implied by the contracted Codazzi and Gauss equations.

A general starting point to attack a pde is local and global well-posedness followed by proving existence and uniqueness of solutions and finally an analysis of the solutions. However, in GR we face a few subtleties. One of them we mentioned above already, namely the general covariance of the Einstein equations. Moreover, the differential structure of the spacetime is not known a priori. Then what should be the ordering on the regions where solutions are defined? One would wish for a domain of dependence theorem to hold globally. (We may think of the wave equation as a simple example to inspire our intuition.) The "magic concept" comes known as *global hyperbolicity* and means that (M, g) admits a Cauchy hypersurface (that is a complete, spacelike hypersurface H in M with each causal curve in the spacetime intersecting H exactly once). Then the *maximal Cauchy development* is given in a unique way as the globally hyperbolic spacetime into which all other such spacetimes imbed isometrically.

Among the active players in the very early years of GR we find D. Hilbert and H. Weyl who contributed substantially to the theory. Let's remind ourselves that in those days the involved mathematical branches had not yet been as developed as today. This caused much confusion about mathematical properties of the Einstein equations and their physical implications. For instance, the pioneers argued about what it means for a solution to behave differently in different coordinate systems. Nowadays, the resolution of these issues is not more than an elegant lemma in geometry. In this context, we understand how important Weyl's "causally connected" world emerged in 1923 hinting at the contents of the domain of dependence theorem which would be established much later. As it turned out, an important tool was introduced by T. de Donder and C. Lanczos and later used by G. Darmois, namely the wave coordinates, as we shall see below. Many people contributed towards a formulation and understanding of the Cauchy problem in GR, we point out also A. Lichnerowicz, K. Stellmacher and K. Friedrichs. On the analysis side, important progress that influenced GR came with the works by H. Lewy, J. Hadamard, J. Schauder and S. Sobolev among many others. In these years, a young woman had made her first and important steps in GR, Yvonne Choquet-Bruhat. She would achieve the big breakthrough in the Cauchy problem in her celebrated works summarized below. As the purpose of the present article does not allow us to delve deeper into the history of the mathematical crescendo of the first half of the 20th century, we refer to Choquet-Bruhat's paper [18] for a more detailed discussion with references of the mathematical progress in GR in these years, whereas the historical facts will be described in her forthcoming autobiography. See also [**?**] for a discussion of the Cauchy problem in view of gravitational waves. Whereas many of the initial problems in GR have been solved, other hurdles have remained tough nuts to crack and bear challenges for future mathematical research.

These are some of the reasons why it took a long time until the Cauchy problem for the Einstein equations was even formulated properly. The breakthrough had to wait until 1952 when Yvonne Choquet-Bruhat [17] proved a local existence and uniqueness theorem for the Einstein equations. And only later the afore-mentioned issues were resolved. In 1953, J. Leray [59] discussed global hyperbolicity. The second breakthrough took place in 1969 when Yvonne Choquet-Bruhat and Robert Geroch [20] proved the global existence of a unique maximal future development for every given initial data set.

We are going to state the fundamental theorems by Choquet-Bruhat and Choquet-Bruhat with Geroch as follows:

Theorem 4 *(Y. Choquet-Bruhat, 1952, [17]) Let $(H, \bar{g}, k)$ be an initial data set satisfying the vacuum constraint equations. Then there exists a spacetime (M, g) satisfying the Einstein vacuum equations with $H \hookrightarrow M$ being a spacelike surface with induced metric $\bar{g}$ and second fundamental form k.*

Theorem 5 *(Y. Choquet-Bruhat, R. Geroch 1969, [20]) Let $(H, \bar{g}, k)$ be an initial data set satisfying the vacuum constraint equations. Then there exists a unique, globally hyperbolic, maximal spacetime (M, g) satisfying the Einstein vacuum equations with $H \hookrightarrow M$ being a Cauchy surface with induced metric $\bar{g}$ and second fundamental form k.*

The main tool in the proof was the use of wave coordinates (often called harmonic coordinates even though the metric is Lorentzian). By definition, wave coordinates x^α satisfy the wave equation

$$\Box_g x^\alpha = 0 \ .$$

This is equivalent to the connection coefficients of these local wave coordinates satisfying

$$g^{mn}\Gamma^\alpha_{mn} = 0 \ .$$

From the fact that the Riemann curvature tensor can be expressed in terms of the connection coefficients, it follows that the EV equations in wave coordinates become

$$\Box_g g_{\alpha\beta} = N_{\alpha\beta}(g, \nabla g) \tag{9}$$

with $N_{\alpha\beta}(g, \nabla g)$ denoting nonlinear terms with quadratics in ∇g. Thus, we have a system of quasilinear wave equations. (9) are the so-called reduced Einstein equations. Choquet-Bruhat in her proof studies the Cauchy problem for this reduced system. Combined with the other main idea, that relies on the domain of dependence theorem, this allowed her to prove theorem 4.

Early analysis of equations of the type like (9) include works by Friedrichs-Lewy and Schauder via energy methods and by Hadamard, Petrovsky and Sobolev by constructing a parametrix.

The above theorems 4, 5 have been generalized to hold for many matter systems. Moreover, improvements were obtained by Dionne, Fisher-Marsden, Hughes-Kato-Marsden using the energy method for initial data given in specific classes of Sobolev spaces. Further improvements followed by Tataru, Smith-Tataru, Klainerman-Rodnianski and Planchon-Rodnianski. Recently, the L^2 curvature conjecture was proven by Klainerman-Rodnianski-Szeftel. The latter show that under certain assumptions the regularity of the data can be relaxed so far that the existence of the solution depends only on the L^2-norms of the Riemannian curvature tensor and on the gradient of the second fundamental form. We only cite the references of the latter, namely [54], [53], [80], [81], [82], [83], please see [54] for a detailed discussion and an extensive list of references.

Characteristic Approach. The characteristic initial value problem for the EV equations (1) starts from initial data given on null hypersurfaces. The data are prescribed on either an outgoing null hypersurface or an ingoing and an outgoing null hypersurface intersecting in a spacelike 2-surface. A. Rendall [73] in 1990 proved the following theorem.

Theorem 6 *(A. Rendall, [73]) Let characteristic smooth initial data for the Einstein vacuum equations be given on null hypersurfaces C_1 and C_2 that intersect transversely on a spacelike surface $S = C_1 \cap C_2$. Then there exists a (non-empty) maximal development (M, g) of the initial data bounded in the past by a neighborhood of S in $C_1 \cup C_2$.*

Rendall's proof reduces the problem to the classic Cauchy problem.

If initial data is given on a single outgoing null hypersurface C, then one has to introduce adapted conditions at the vertex o of C for the solution. To establish his main result, Christodoulou considers data which is trivial up to a surface S. More precisely, in the context of Rendall's theorem 6 this corresponds to C to the future of S being C_1, whereas C_2 being the incoming Minkowski cone $\underline{C}$ rooting in S.

A basic difference between the classic and the characteristic initial value problem is that in the former the constraints for the initial data are given by elliptic pde, whereas in the latter they can be written as ode (ordinary differential equations). That is, we can specify some data freely and solve propagation equations along the generators of null hypersurfaces. Thus, this aspect is much simpler in the characteristic situation.

The characteristic treatment is more natural for questions concerning gravitational radiation: Firstly, in the investigation of black hole formation through the focussing of gravitational waves, as is the main present topic. Secondly, to analyze gravitational waves coming from sources like the mergers of black holes. The reason being that these waves travel at the speed of light along null hypersurfaces of the spacetime.

Stability of Minkowski Space. Considering asymptotically flat systems under gravitation, we would like to understand under which conditions there exist global solutions of a certain smoothness and what their structures are, respectively when do singularities (black holes) form. The above theorems do not answer these questions but constitute the way to start. The major breakthrough [27] was achieved in 1993 by Christodoulou and Klainerman proving that for asymptotically flat initial data being small in weighted Sobolev spaces there exists a complete maximal development as a solution of the EV equations (1). This is known as the global nonlinear stability of the Minkwoski space. A summarized version of their theorem reads as follows:

Theorem 7 *(D. Christodoulou and S. Klainerman, 1993, [27]) Let be given strongly asymptotically flat initial data for the EV equations (1) being sufficiently small. Then there exists a unique, causally geodesically complete and globally hyperbolic solution (M, g), that itself is globally asymptotically flat.*

The geometric-analytic proof is monumental, does not depend on coordinates and lays open the structures of the solution spacetimes. In a first part, suitable energies are identified in the Bel-Robinson tensor. The latter basically is a quadratic of the Weyl tensor and is used heavily also in Christodoulou's work [24]. We will give the formula below. Next, the curvature components are estimated from these energies via a comparison argument. Then, in the main part of the proof, constituting a large bootstrap argument, under assumptions on the curvature it is shown that the remaining geometric quantities are controlled. Many of the new features and ideas in the proof have had impact far beyond GR in the study of other nonlinear hyperbolic pde. See also the semi-global result by Friedrich [43], later proofs under more assumptions and using wave coordinates by Lindblad and Rodnianski [61], [62], as well as the proof for the exterior part with a double null foliation by Klainerman and Nicolò [49]. The Christodoulou-Klainerman result [27] was generalized in 2000 by N. Zipser, [93], [94], for the Einstein-Maxwell system and in 2007 by L. Bieri, [6], [7], for the EV equations with less assumptions on the decay at infinity and less regularity thereby establishing the borderline case for decay of the data in the EV situation. The proofs in both these works are geometric-analytic.

A specific feature in the proof of [27] turns out to be crucial not only in order to establish theorem 7 but also in Christodoulou's new constructions in [24]: The above foliation into null hypersurfaces C_u is not arbitrary, but depends on a suitably chosen optical function u, whereas the spacelike hypersurfaces H_t are generated by a foliating maximal time function t. It follows from the proof in [27] that null infinity $\mathcal{I}^+$ is complete for the data considered, however, what is known as "peeling" does not hold for all the curvature components. To understand the latter, let us focus for a moment on radiation, which we introduced as fluctuation of the curvature. Thus, in order to investigate gravitational waves coming from far-away sources, we need to determine the properties of the curvature components at null infinity. These components are obtained through contraction with vector fields of a null frame given by the above foliations. As the waves propagate at the speed of light (and light travels along null geodesics), we have to follow them along outgoing null geodesics. This type of questions was addressed already around the 1960s. Trautman [88], Bondi [13], Bondi-van-der-Burg-Metzner [14], Sachs [77], Penrose [65] pioneered the use of null hypersurfaces to describe gravitational radiation. Other discussions were given by Pirani [72], Newman and Penrose [64], Geroch [45], Ashtekar and Hansen [3], Ashtekar and Schmidt [4], Ashtekar and Streubel [5].

All these works address gravitational radiation in some way. One of the problems in studying $\mathcal{I}^+$ arises when (as in some of the cited papers) one would like to expand the metric in power series in r^{-1} with coefficients depending on u and the angular coordinates. Or in general we can ask: How smooth should null infinity be? In the later cited papers, the authors replaced the assumptions about the power series expansion by another assumption which also requires a minimal regularity. Thus, if one conformally compactifies the boundary at null infinity, this implies a minimal regularity of the data, which in the aforementioned works would be at least C^2. However, Christodoulou showed that for physical spacetimes C^2 is impossible. In fact, in the general case considered here, the conformal factor extends to $\mathcal{I}^+$ as a function in $C^{1,\alpha}$. The works by Christodoulou and Klainerman

[27] are within that regime. From the smoothness follows a specific hierarchy of decay for the curvature components, which is called peeling. Today, we mainly refer to the stronger assumptions as the *Newman-Penrose picture* and to the later more general situation as the *Christodoulou-Klainerman picture*.

The above foliation developed in [27] is natural in the way that one follows the waves along the C_u. In the corresponding null-frame the structure equations relate curvature components with the connection coefficients. A particularly interesting representative of the latter in view of radiation is the shear $\hat{\chi}$ which satisfies

$$\not{div}\,\hat{\chi} = \frac{1}{2}(\not{\nabla}\, tr\chi + \zeta tr\chi) - \hat{\chi} \cdot \zeta - \beta$$

where slashed quantities are on the surfaces $S_{t,u}$, β denotes a curvature component and ζ, the torsion 1-form, is another connection coefficient. These equations lay open structures of the spacetime that had previously been inaccessible, in particular could not be captured by the corresponding equations of the Newman-Penrose formalism. In [27] the authors derived thereby a new method to treat the Cauchy problem for the Einstein equations (being hyperbolic), coupling elliptic equations for $\hat{\chi}$ and related quantities on $S_{t,u}$ with propagation equations along the null hypersurfaces C_u. The coupling term for the above elliptic system being $\not{\nabla}\, tr\chi$ the propagation equation along the generators of C_u reads (above and here we omit the indices for simplicity)

$$\frac{\partial \not{\nabla}\, tr\chi}{\partial s} + tr\chi \not{\nabla}\, tr\chi + 2\hat{\chi}\not{\nabla}\,\hat{\chi} = 0\ .$$

This kind of structure plays an important role as well in [24]. As explained in the sketch of the proof of theorem 7 energy estimates for the curvature precede the handling of $\hat{\chi}$ and related quantities. In a bootstrap argument with control on the curvature, the connection coefficients are estimated.

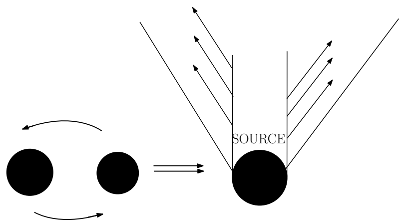
In Christodoulou's monumental work on black hole formation, incoming gravitational waves follow $C_{\underline{u}}$. This will be treated in chapter 4. Latest works on stability of black holes is addressed in chapter 5. Before concentrating on ingoing waves however, in the next chapter we say a few words about outgoing gravitational waves that were detected by LIGO [1] in 2015.

3 Gravitational Radiation

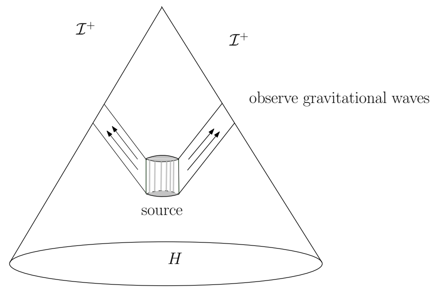
3.1 Gravitational Waves

Gravitational waves are fluctuations of the spacetime curvature traveling at the speed of light along null hypersurfaces C_u. For the first time, LIGO [1] detected such waves in 2015. This constitutes the beginning of a new era in science where these waves through LIGO and other detectors will reveal information from to date unknown regions of the universe. Thereby, the mathematical understanding of the dynamics of the Einstein equations, in particular the Cauchy problem, will be crucial.

A typical source for gravitational waves is the merger of two black holes. They will spiral in and finally merge thereby radiating away energy in form of gravitational waves.



Black holes spiral in and finally merge, sending out gravitational waves.



Gravitational radiation: gravitational waves traveling from source along outgoing null hypersurfaces.

Let us consider future null infinity $\mathcal{I}^+$ defined above and the Christodoulou-Klainerman result of theorem 7, but now we omit the smallness assumptions and start with large initial data that might even contain black holes. More precisely, let $(H_0, \bar{g}, k)$ be an arbitrary asymptotically flat initial data set, only vacuum outside a compact set. By the domain of dependence theorem and the results of [27] one can show that in the new situation we can still attach a piece of asymptotic boundary $\mathcal{I}^+$ to the Cauchy development of the initial data, $\mathcal{I}^+$ being parametrized by $(-\infty, u_+) \times S^2$ as opposed to $(-\infty, +\infty) \times S^2$ from before. Thus, limits along appropriate null hypersurfaces C_u can be computed even in the new case. This follows also from [25], [49].

Gravitational radiation is described on $\mathcal{I}^+$. Most important are the limits of the shears $\underline{\hat{\chi}}$, $\hat{\chi}$ and the curvature component $\underline{\alpha}$ which is contracted twice with the incoming null vectorfield $\underline{L}$ and decaying like r^{-1}. The radiative amplitude per unit solid angle is given by the following limit

$$\Xi(u, \theta) = \lim_{C_u, t \to \infty} r \underline{\hat{\chi}} \tag{10}$$

being a symmetric, traceless 2-tensor. Similarly, one defines

$$\Sigma(u, \theta) = \lim_{C_u, t \to \infty} r^2 \hat{\chi} \ . \tag{11}$$

In the last chapter of [27] these limits and the asymptotic structures of the spacetimes are derived. The following crucial relations emerge from these studies:

$$\frac{\partial \Sigma}{\partial u} = -\Xi \tag{12}$$

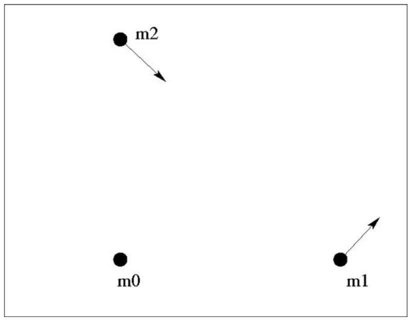
$$\frac{\partial \Xi}{\partial u} = -\frac{1}{4} A \tag{13}$$

with A denoting the limit of the curvature component $\underline{\alpha}$:

$$A(u, \theta) = \lim_{C_u, t \to \infty} r \underline{\alpha} \ . \tag{14}$$

It is interesting to see that the intricate local structures of radiative spacetimes evolve into a simpler picture at null infinity $\mathcal{I}^+$. This is then directly related to gravitational wave experiments. Gravitational radiation changes the spacetime while traveling through.

Doing a gravitational wave experiment, we can think of ourselves (and the detector) sitting at null infinity $\mathcal{I}^+$ as the waves are coming from sources very far away. In fact, we identify our position at retarded time u_* as $(u_*, \theta) \in \mathcal{I}^+$ and evolving in u. A detector like LIGO consists of 3 test masses m_0, m_1, m_2 suspended by pendulums (or floating on their geodesics if the experiment is done in space). These masses are located in an L-shape as in the following picture and are at large distance r and angular direction θ from the source. By laser interferometry the distances of m_1 and m_2 with respect to the reference test mass m_0 are measured. The u-rate of change of this relative displacement is determined by $\Xi(u, \theta)$.



These test masses follow geodesics and for an experiment on Earth like LIGO their relative acceleration is expressed through curvature in the Jacobi equation:

$$\nabla_U^2 V = R(U, V)V \tag{15}$$

where U denotes the tangent vector for an object in free fall separated from a second object by a vector V.

3.2 Memory Effect of Gravitational Waves

So far, we have thought of gravitational waves in a region of the spacetime (M, g) as changing that region "instantaneously" while traveling through. However, there is more to the story. It is predicted that gravitational radiation permanently changes the spacetime, leaving a footprint in the regions it passes. This is called the *memory effect of gravitational waves* today known as the *Christodoulou effect*. In 1974, Ya.B. Zel'dovich and A.G. Polnarev found [92] such an effect for the linearized Einstein equations. However, it was believed to be too small for detection. In 1991, D. Christodoulou [22] investigated the full Einstein equations and derived a memory that was larger than the one of the previous work. In fact, L. Bieri and D. Garfinkle showed [11] that these are two different effects. For memory, see also the following works by Braginsky and Grishchuk [15], Thorne [16], Blanchet and Damour [30], and more recent Bieri, Chen and S.-T. Yau [8], [9], Bieri and Garfinkle [10], [11], Wald and Tolish [87], Flanagan and Nichols [41], [42], Favata [39]. We refer to these articles for further references.

To explain how this effect can be measured we consider the Jacobi equation (15). In the above figure showing the 3 test masses the arrows refer to a permanent displacement in the horizontal plane. For simplicity of the discussion, assume that the wave source is perpendicular to the plane of the 3 test masses. Equation (15) gives an acceleration on the left hand side and curvature on the right hand side. Here is where the two relations ((12),(13)) play a crucial role. First using (13) in (15) where the leading order curvature term is A, and integrating once, second using (12) to substitute the shear terms and integrate again, finally taking the limit as $u \to \infty$, we obtain an equation of the following schematic form (omitting indices):

$$\triangle x = C(\Sigma^+ - \Sigma^-) \tag{16}$$

where $\Sigma^{\stackrel{+}{-}}$ denotes the limits of Σ when $u \to +\infty$ respectively $u \to -\infty$ and $\triangle x$ is the distance of the permanent relative displacement, C a particular constant divided by r. It can be shown by geometric-analytic investigations that $\Sigma^+ - \Sigma^-$ is related to

$$F = \mathcal{C} \int_{-\infty}^{+\infty} |\Xi|^2 du \tag{17}$$

with $\frac{F}{4\pi}$ being the total energy radiated away in a given direction per unit solid angle. An experiment how to detect gravitational wave memory with Advanced LIGO has recently been suggested in [56] by P. Lasky, E. Thrane, Y. Levin, J. Blackman and Y. Chen.

4 The Formation of Closed Trapped Surfaces

4.1 Precise Formulation of the Result: Initial Data and Evolution

Let us consider incoming gravitational waves concentrating in a small region. The claim of theorem 3 by Christodoulou is that if the amount of energy is above a certain threshold, then a closed trapped surface will form. We are now going to state this result in a more precise way, explain the setting and discuss the main ideas of the monumental proof.

We start by considering a spacetime manifold (M, g) with boundary, being a smooth solution of the Einstein vacuum equations (1) such that the past boundary of M is the future null geodesic cone C_0 of a point O. Initial data is given on C_0, assuming that it is trivial in a neighborhood of O, and our (M, g) is to be a development of this initial data. (See figures after theorem 10.) We introduce T to be a unit future-directed timelike vector T at the vertex O and denote by γ_0 the geodesic generated by T and such that the tangent vector at O to each null geodesic generator has projection T along T. The generators of the cone C_0 are parametrized by an affine parameter s measured from O and such that s is the parameter of the geodesic null vectorfield L' along C_0 being the tangent field of each generator. Now, the initial data is assumed to be trivial for $s \leq r_0$ for some $r_0 > 1$. This means that they coincide with the data corresponding to a truncated cone in Minkowski spacetime. Then the boundary of this region with trivial data is a round sphere of radius r_0. In particular, each generator extends up to parameter value $r_0 + \delta$ where δ is a constant $1 \geq \delta > 0$. It is assumed that C_0 does not contain any conjugate or cut points.

The domain of dependence theorem guarantees that the solution spacetime has a region that is Minkowskian and that is given by the past of a backwards light cone $\underline{C}_e$ of a point e on γ_0 at distance $2r_0$ from O.

An advanced time function $\underline{u}$ on C_0 is defined by

$$\underline{u} = s - r_0 \ . \tag{18}$$

In the proof, the spacetime will be constructed from the initial data whereby the level sets $C_{\underline{u}}$ of $\underline{u}$ are required to be ingoing null hypersurfaces.

In analogy to the previous sections, we denote by $\hat{\chi}$ the trace-free part of χ which is the second fundamental form of the sections S_s of C_0 corresponding to constant values of the affine parameter s and let $\not{g}$ be the induced metric on S_s. Thus, $\hat{\chi}$ is the shear of these sections.

We introduce the following crucial function:

$$e = \frac{1}{2} |\hat{\chi}|^2_{\not{g}} \ . \tag{19}$$

This e is an invariant of the conformal intrinsic geometry of C_0.

With these tools, we now state the following version of the main theorem:

Theorem 8 (Closed Trapped Surface Formation). (D. Christodoulou [24]) *Let k, l be positive constants such that $k > 1 > l$. Let initial data be given as described above and assume that*

$$\frac{r_0^2}{8\pi}\int_0^\delta e \; d\underline{u} \;\geq\; \frac{k}{8\pi} \tag{20}$$

with the integral along C_0 where $\underline{u} \in [0, \delta]$ for some $\delta > 0$. Then, if δ is suitably small, the maximal development of the data contains a closed trapped surface S diffeomorphic to S^2 and has area

$$Area(S) \;\geq\; 4\pi l^2 \;. \tag{21}$$

The notion in theorem 3 of incoming energy per unit solid angle in each direction in a suitably small time interval is replaced in theorem 8 by the left hand side of (20). The reason is that the incoming energy per unit solid angle in each direction in the advanced time interval $[0, \delta]$ is only defined at past null infinity. Thus, from theorem 8 Christodoulou derives another result with initial data given at past null infinity, formulated for the moment in theorem 9 below. We will analyze the details of 9 in the next sections. For these investigations one has to let $r_0 \to \infty$ and thereby move C_0 back to past null infinity $\mathcal{I}_-$. From our previous discussions about radiation at future null infinity $\mathcal{I}_+$ it is straightforward to see that the limit of the rescaled quantity of e corresponds to the analogue of $|\Xi|^2$ defined through (10), and because we are at past null infinity, we have to replace $\underline{\hat{\chi}}$ by $\hat{\chi}$.

Theorem 9 (Closed Trapped Surface Formation). (D. Christodoulou [24]) *Let k, l be constants as in theorem 8. Let smooth asymptotic initial data be given at past null infinity $\mathcal{I}_-$ being trivial for $\underline{u} \leq 0$. Assume that the incoming energy per unit solid angle in each direction in the advanced time interval $[0, \delta]$ is greater or equal than $\frac{k}{8\pi}$. Then, if δ is suitably small, the maximal development of the data contains a closed trapped surface S diffeomorphic to S^2 and has area*

$$Area(S) \;\geq\; 4\pi l^2 \;. \tag{22}$$

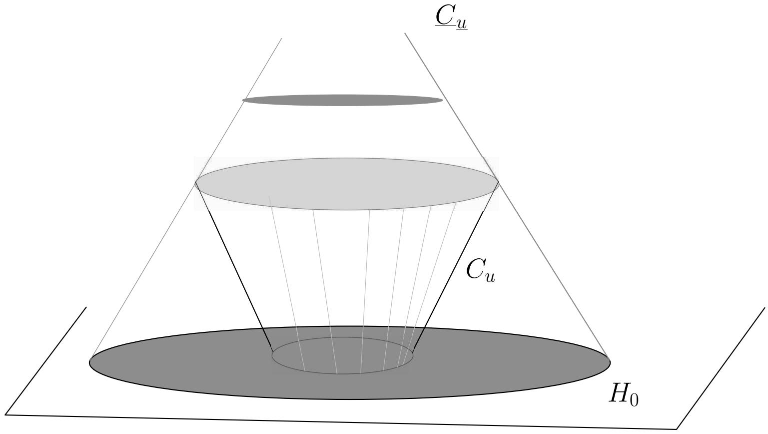
4.2 The Optical Structure

Double Null Foliation: This foliation relies on the two optical functions u and $\underline{u}$. In (18) we defined $\underline{u}$. We emphasize that for each value v the corresponding level set of $\underline{u}$ is the incoming null hypersurface $\underline{C}_v$. Now, the function u conjugate to $\underline{u}$ is introduced such that for each v the v-level set of u is the outgoing null hypersurface C_v. The notation hereafter for these outgoing null hypersurfaces is C_u and for the incoming ones $\underline{C}_{\underline{u}}$. We note that the outgoing level sets C_u of u emanate from points on γ_0, and that $u|_{\gamma_0}$ measures arc length from O along γ_0 minus r_0.

The intersections

$$S_{\underline{u},u} = \underline{C}_{\underline{u}} \cap C_u \tag{23}$$

are spacelike 2-surfaces.



Furthermore, we define the H_t by

$$u + \underline{u} = t \ .$$

It is clear that we do not expect this foliation to exist for very long. Though, it will exist up to a null hypersurface $\underline{C}_\delta$ and a hypersurface H_c for small δ and for $u_0 < c < 0$. We denote this region by M_R.

The proof of the main theorem above will rely on a continuity argument: Regarding (M, g) we suppose that the generators of $\underline{C}_{\underline{u}}$ and C_u have no end points in $M \backslash \gamma_0$. This means in particular that the C_u do not contain any conjugate or cut points in M and that the $\underline{C}_{\underline{u}}$ do not contain any focal or cut points in $M \backslash \gamma_0$. Then it follows that in $M_R \backslash \gamma_0$ the $\underline{C}_{\underline{u}}$ and C_u are smooth, H_t is a spacelike hypersurface and the spacelike 2-surfaces $S_{\underline{u},u}$ are embedded in $M_R \backslash \gamma_0$. (We note that the $S_{\underline{u},u}$ are diffeomorphic to S^2.)

The optical functions u and $\underline{u}$ themselves obey the eikonal equation:

$$\begin{aligned} (g^{-1})^{\mu\nu} \partial_\mu u \partial_\nu u &= 0 \\ (g^{-1})^{\mu\nu} \partial_\mu \underline{u} \partial_\nu \underline{u} &= 0 \ . \end{aligned}$$

Next, we are going to introduce three *null frames* related to this structure.

First, the future-directed null geodesic vectorfields L' and $\underline{L}'$ are given by

$$L'^\mu = -2(g^{-1})^{\mu\nu} \partial_\nu u \ \ , \ \ \underline{L}'^\mu = -2(g^{-1})^{\mu\nu} \partial_\nu \underline{u} \ . \tag{24}$$

From this is defined a positive function Ω as follows:

$$-g(L', \underline{L}) = 2\Omega^{-2} \ .$$

We observe that Ω is the inverse density of the double null foliation.

Second, the normalized vectorfields $\hat{L}$ and $\hat{\underline{L}}$ are defined as

$$\hat{L} = \Omega L' \ \ , \ \ \hat{\underline{L}} = \Omega \underline{L}' \ . \tag{25}$$

They satisfy

$$g(\hat{L}, \hat{\underline{L}}) = -2 \ .$$

Third, we define the vectorfields L and $\underline{L}$ as

$$L = \Omega^2 L' \ \ , \ \ \underline{L} = \Omega^2 \underline{L}' \ . \tag{26}$$

They satisfy

$$\begin{aligned} Lu = &\quad 0 \quad = \underline{L}\underline{u} \\ L\underline{u} = &\quad 1 \quad = \underline{L}u \ . \end{aligned}$$

We find that the integral curves of L are the generators of the outgoing null hypersurfaces C_u parametrized by $\underline{u}$ and that the integral curves of $\underline{L}$ are the generators of the incoming null hypersurfaces $\underline{C}_{\underline{u}}$ parametrized by u. Following the notation of [24] we define the flow Φ_τ generated by L on any C_u and the flow $\underline{\Phi}_\tau$ generated by $\underline{L}$ on any $\underline{C}_{\underline{u}}$. The $\Phi_\tau : S_{\underline{u},u} \to S_{\underline{u}+\tau,u}$ and $\underline{\Phi}_\tau : S_{\underline{u},u} \to S_{\underline{u},u+\tau}$ are diffeomorphisms.

Thus, the above structures yield *canonical coordinate systems*. Let (θ^1, θ^2) be local coordinates on a domain U on S_{0,u_0}. Then we can extend these to $\underline{\Phi}_u(\Phi_{\underline{u}}(U)) \subset S_{\underline{u},u+u_0}$. Now, given two domains U_1 and U_2 with coordinates $(\theta^A; A = 1,2)$ respectively $(\theta'^A : A = 1,2)$ covering S_{0,u_0}, $M_R \backslash \gamma_0$ is covered by two regions with coordinates $(\underline{u}, u, \theta^A : A = 1,2)$ and $(\underline{u}, u, \theta'^A : A = 1,2)$, respectively. $(\underline{u}, u) \in D \backslash \gamma_0$. The domain D is depicted in the next figure right after theorem 10.

The vectorfields $\underline{L}$ and L in these canonical coordinates read as

$$\underline{L} = \frac{\partial}{\partial u} \quad , \quad L = \frac{\partial}{\partial \underline{u}} + b^A \frac{\partial}{\partial \theta^A}$$

with b^A obeying

$$\frac{\partial b^A}{\partial u} = 4\Omega^2 \zeta^{\#A}$$

for the torsion ζ. (It is $\zeta^{\#A} = (\not{g}^{\,-1})^{AB}\zeta_B$ and $\zeta_A = \zeta(\frac{\partial}{\partial \theta^A})$.) Finally, the metric g in these coordinates is given by

$$g = -2\Omega^2 (du \otimes d\underline{u} + d\underline{u} \otimes du) + \not{g}_{AB} (d\theta^A - b^A d\underline{u}) \otimes (d\theta^B - b^B d\underline{u}) \ . \tag{27}$$

It turns out that stereographic coordinates on the sphere are especially nice to work with in this problem.

4.3 "The" Theorems

The overarching principle of the *proof of black hole formation* is two-fold: First, the spacetime has to be constructed and it has to be shown that it exists long enough such that a closed trapped surface can form. This solution has to be "nice" enough (i.e. sufficiently smooth, no focal points for instance). Second, the formation of a trapped surface has to be proven.

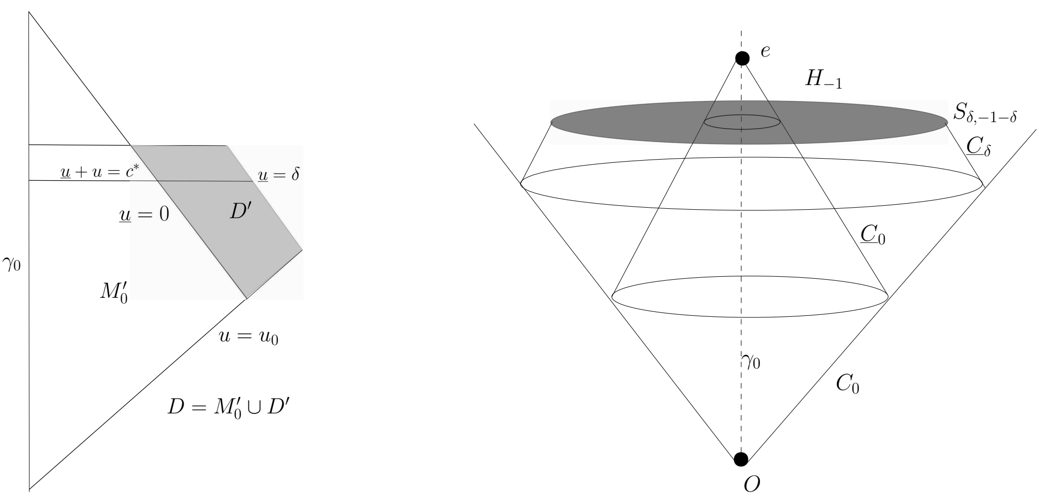
These are formulated in *"the" two main theorems*, namely the *existence* theorem 10 and the *closed trapped surface formation* theorem 8.

The former is much more difficult and subtle to establish than the latter. Therefore, the sketch of the proof of theorem 10 will take most of the remaining part of this article whereas theorem 8 is shown in a more straightforward manner.

In order to state the existence theorem 10 we briefly revisit the initial data from section 4.1. There we give initial data on a complete future null geodesic cone C_0, the data being trivial for $s \leq r_0$. We consider the restriction of the initial data to $s \leq r_0 + \delta$, thus restricting to $\underline{u} \in [0, \delta]$ where the data is trivial for $\underline{u} \leq 0$. At the vertex O and thereby on C_{u_0} we set u equal to $u_0 = -r_0$. As $2(u - u_0)$ along γ_0 equals the arc length from O, our u is determined everywhere. Therefore, the spacetime that we want to construct will be bounded in the future by H_{-1} where $\underline{u} + u = -1$, and by $\underline{C}_\delta$. Following the notation of [24] we call this development M_{-1}, and call M'_{-1} the non-trivial region of M_{-1} given by $\underline{u} > 0$.

The existence theorem is stated as follows.

Theorem 10 (Existence). (D. Christodoulou [24]) *Let initial data be given as described in section 4.1 and in the previous paragraph. Let δ be sufficiently small. Then the maximal development under the Einstein vacuum equations (1) contains a region M_{-1} where the foliation given in section 4.2 can be constructed such that the null hypersurfaces C_u and $\underline{C}_{\underline{u}}$ contain no cut or conjugate points. M_{-1} is bounded in the future by the spacelike hypersurface H_{-1} and the incoming null hypersurface $\underline{C}_\delta$.*



This existence theorem not only establishes the solution but it also yields important information on the geometric and analytic structures of these solutions, thus on spacetimes where closed trapped surfaces begin to form.

4.4 Proof of the Existence Theorem

Before we start discussing the essentials of the proof, let us ask the following question: Knowing the energy method from pde theory, does there exist something similar for the Einstein equations? Yes, there is a generalized form of this idea as we mentioned already above in the ideas of the proof of theorem 7. As an ultra short summary we may recall: energy controls curvature which controls the other geometric quantities in a bootstrap argument. Of course, interesting structures are hiding behind these concepts that have to be unraveled. It is important that the energies are constructed with respect to "useful" vectorfields. In the following outline of the main points of the proof of theorem 10 we are going to investigate these structures and connect them with new features of the problem under study.

In this section, we give a sketch of the main methods and ideas of the proof of the existence theorem 10. While two of these methods were introduced by Christodoulou and Klainerman in [27], the third method was developed by Christodoulou in [24].

One of the methods from [27] is specific to the Einstein equations, whereas the other can be applied to a broad range of nonlinear hyperbolic pde, in particular to all Euler-Lagrange systems of hyperbolic pde. The first of these methods concerns the Bianchi identities (3) with the goal to obtain estimates for the spacetime curvature W. We remark that in an Einstein vacuum spacetime (M, g) the curvature $R_{\alpha\beta\gamma\delta}$ is equal to the Weyl curvature $W_{\alpha\beta\gamma\delta}$. (This is clear from the fact that the Riemannian curvature $R_{\alpha\beta\gamma\delta}$ splits into its traceless part $W_{\alpha\beta\gamma\delta}$ and a part containing the Ricci and scalar curvature which are zero by virtue of the EV equations (1).) More generally, a *Weyl field* W on (M, g) is a 4-covariant tensorfield with the algebraic properties of the Weyl or conformal curvature tensor. Left ${}^*W$ and right W^* Hodge duals of W can be defined and it is shown that ${}^*W = W^*$. There is a nice analogy with Maxwell's theory of electromagnetism. For, a Weyl field satisfies equations that are similar to Maxwell's equations for an electromagnetic field. The *Bianchi equations* for W are the following linear equations:

$$\nabla^\alpha W_{\alpha\beta\gamma\delta} = J_{\beta\gamma\delta} \tag{28}$$

with $J_{\beta\gamma\delta}$ being a *Weyl current*. Moreover, the following holds

$$\nabla_{[\alpha} W_{\beta\gamma]\delta\epsilon} = \epsilon_{\mu\alpha\beta\gamma} J^{*\mu}{}_{\delta\epsilon}$$

ϵ denoting the volume form for of the spacetime manifold and $J^*_{\beta\gamma\delta} = \frac{1}{2} J_\beta{}^{\mu\nu} \epsilon_{\mu\nu\gamma\delta}$. If the Weyl field W is the spacetime curvature itself, then the corresponding Weyl current vanishes and we are back to the Bianchi identities (3).

Let us define the deformation tensor ${}^{(Y)}\hat{\pi}$ of Y to be the trace-free part of $\mathcal{L}_Y g$. We may think of this quantity to measure how the conformal geometry of (M, g) changes under the flow of Y.

We have to deal with more general Weyl fields. And as the Lie derivative $\mathcal{L}_Y W$ of a Weyl field is in general not a Weyl field, because it has trace, Christodoulou and Klainerman introduce the modified Lie derivative $\hat{\mathcal{L}}_Y W$, which is trace-free and has all the other properties of a Weyl field, thus it is a Weyl field. Similarly, this holds for J. Due to certain conformal covariance properties of the Bianchi equations it follows that the Weyl current associated to $\hat{\mathcal{L}}_Y W$ is the sum of $\hat{\mathcal{L}}_Y J$ and a bilinear term that is itself linear in ${}^{(L)}\hat{\pi}$ and its first covariant derivative and in W and its first covariant derivative.

New Weyl fields are generated from the original curvature tensor of (M, g) by consecutively applying $\hat{\mathcal{L}}_{Y_i}$ for $i = 1, \cdots, n$ where $Y_1, \cdots, Y_n$ are *commutation vectorfields.*

In order to make use of some form of the energy method for the Einstein equations we introduce the Bel-Robinson tensor $Q(W)$ associated to W. This Q plays a role similar to the energy-momentum-stress tensor T for the electromagnetic field.

$$Q_{\alpha\beta\gamma\delta} \ = \ \frac{1}{2} \left(W_{\alpha\rho\gamma\sigma} \, W_\beta{}^\rho{}_\delta{}^\sigma \ + \ {}^*W_{\alpha\rho\gamma\sigma} \ {}^*W_\beta{}^\rho{}_\delta{}^\sigma \right) . \tag{29}$$

It satisfies the following positivity condition:

$$Q \ (X_1, \ X_2, \ X_3, \ X_4) \ \geq \ 0 \tag{30}$$

where X_1, X_2, X_3 and X_4 are future-directed causal vectors. Q is symmetric and trace-free in any pair of indices. Moreover, if W satisfies the Bianchi equations then Q is divergence-free:

$$D^\alpha \ Q_{\alpha\beta\gamma\delta} \ = \ 0 \ . \tag{31}$$

In general, $divQ$ equals an expression linear in W and in J.

From Q for a given W we define the *energy-momentum density* vectorfield $P(W; X_1, X_2, X_3)^\alpha$ associated to W and to the three *multiplier vectorfields* X_1, X_2, X_3, which are future-directed causal:

$$P(W; X_1, X_2, X_3)^\alpha = -Q(W)^\alpha{}_{\beta\gamma\delta} X_1^\beta X_2^\gamma X_3^\delta \ . \tag{32}$$

It follows that $divP$ equals the sum of $-(divQ(W))(X_1, X_2, X_3)$ and a bilinear expression that is linear in $Q(W)$ and in ${}^{(X_1)}\hat{\pi}$, ${}^{(X_2)}\hat{\pi}$, ${}^{(X_3)}\hat{\pi}$.

The divergence theorem in spacetime together with the positivity property of $Q(W)$ yield control of all the derivatives of the curvature up to required order m. This latter control is realized via the integrals on the future boundary.

The second method from [27] (with wide applications) used in [24] is tightly interwoven with the first one and roots in the specific foliations of the spacetime (discussed above). Whereas the optical function u lay open the structures of the natural flow of outgoing gravitational waves along the C_u and which played a crucial role in [27] and [22], the optical function $\underline{u}$ is most crucial in [24] because it follows incoming gravitational radiation along the $\underline{C}_{\underline{u}}$. The trapped surfaces that form in the evolution in [24] are sections $S_{\underline{u},u}$ given in (23) of "late" C_u everywhere along which precisely these C_u have negative expansion. These structures also allow us to identify natural vectorfields that are used within the energy method. One requirement is that the set of commutation vectorfields has to span the tangent space to M at each point.

Finally, the third and newest method in Christodoulou's [24] is the so-called *short pulse method*. It has a wide range of applications in other nonlinear pde. It is a specific way of treating the focusing of incoming waves. A main feature is, that initial data has to be sufficiently *large* so that a closed trapped surface will form. In a broader context, one can consider Euler-Lagrange systems of (nonlinear) hyperbolic pde and establish an existence theorem for large initial data and study the evolution to understand interesting phenomena that may occur.

We recall that a heuristic version of the main theorem above includes "large" incoming energy during a "small" time interval. Thus, there is a small parameter involved that we call δ. In what follows we will analyze the role of this parameter in the data thereby explaining the short pulse method.

After a technical setup, one starts with the simple task to analyze the equations along C_{u_0}. As C_{u_0} is a null hypersurface and one faces a characteristic initial value problem, one can prescribe free data not worrying about constraints. Then the full set of data (including all the curvature components and their transversal derivatives) is easily obtained by integrating ode along the generators of C_{u_0}.

The free data can be given as 2-covariant, symmetric, positive definite tensor density m on the sphere and depending on $\underline{u}$, moreover m being of weight -1 and $\det m = 1$. In particular, we write

$$m = \exp \psi \tag{33}$$

with $\psi \in \hat{S}$ the latter denoting the space of symmetric, trace-free, 2-dimensional matrices. Thus, $\exp : \hat{S} \to H_1^+$ is an analytic diffeomorphism. The transformation rule turns out to be especially simple for stereographic charts on S^2.

For the short pulse ansatz in [24] Christodoulou considers an arbitrary, 2-dimensional, smooth, symmetric, trace-free matrix-valued map ψ_0 on S^2 that depends on $s \in [0, 1]$ and that extends smoothly to $s \leq 0$. Introduce the following:

$$\psi(\underline{u}, \theta) = \frac{\delta^{\frac{1}{2}}}{|u_0|} \psi_0(\frac{\underline{u}}{\delta}, \theta) \quad , \quad (\underline{u}, \delta) \in [0, \delta] \times S^2 \quad . \tag{34}$$

Then the equations along C_0 are analyzed and yield a specific structure of the spacetime curvature along C_{u_0}. In order to state these, we decompose the spacetime curvature $R_{\alpha\beta\gamma\delta}$ with respect to the natural foliation discussed above. This yields for any vectors $X, Y \in$

$T_P S_{\underline{u},u}$ at a point p:

$$\begin{aligned}
\alpha(X,Y) &= R(X,\hat{L},Y,\hat{L}) \\
\underline{\alpha}(X,Y) &= R(X,\hat{\underline{L}},Y,\hat{\underline{L}}) \\
\beta(X) &= \frac{1}{2}R(X,\hat{L},\hat{\underline{L}},\hat{L}) \\
\underline{\beta}(X) &= \frac{1}{2}R(X,\hat{\underline{L}},\hat{\underline{L}},\hat{L}) \\
\rho &= \frac{1}{4}R(\hat{\underline{L}},\hat{L},\hat{\underline{L}},\hat{L}) \\
\sigma\not\in(X,Y) &= \frac{1}{2}R(X,Y,\hat{\underline{L}},\hat{L})
\end{aligned}$$

where $\not\in$ denotes the area form of $S_{\underline{u},u}$. We note that $\alpha, \underline{\alpha}$ are symmetric, 2-covariant, trace-free tensorfields on $S_{\underline{u},u}$, whereas $\beta, \underline{\beta}$ are 1-forms on $S_{\underline{u},u}$ and ρ, σ are functions on these surfaces. The following structures are obtained for these components along C_{u_0}:

$$\begin{aligned}
\sup_{C_{u_0}} |\alpha| &\leq O_2(\delta^{-\frac{3}{2}}|u_0|^{-1}) \\
\sup_{C_{u_0}} |\beta| &\leq O_2(\delta^{-\frac{1}{2}}|u_0|^{-2}) \\
\sup_{C_{u_0}} |\rho| \ , \ \sup_{C_{u_0}} |\sigma| &\leq O_3(|u_0|^{-3}) \\
\sup_{C_{u_0}} |\underline{\beta}| &\leq O_4(\delta|u_0|^{-4}) \\
\sup_{C_{u_0}} |\underline{\alpha}| &\leq O_5(\delta^{\frac{3}{2}}|u_0|^{-5})
\end{aligned} \tag{35}$$

Pointwise quantities are taken with respect to the induced metric $\not{g}$ on the 2-surfaces. [2]

One of the deep insights of the work [24] is disclosed on the right hand side of (35): This particular dependence on δ of the curvature components is called the *short pulse hierarchy*. Indeed, we read off a *nonlinear* hierarchy. (A direct computation shows that the linearized equations would give a different hierarchy.)

With the ansatz (34) one observes that the amplitude of the pulse is proportional to the square root of the pulse length. This interesting relation only shows in the nonlinear theory, it does not exist in a linear one. A closer look makes evident that this hierarchy is required for trapped surfaces to form in M_{-1}. Thus, it is the heart of the proof of the existence theorem 10.

The main challenge of this method is to prove that the specific hierarchy is preserved in the evolution. Towards this goal, vectorfields with specific weights are chosen in connection with the energies and currents defined above. In particular, as multiplier vectorfields we take L and K with

$$K = u^2 \underline{L} \ .$$

Then for each Weyl field (curvature and modified Lie derivatives of curvature) one defines energy-momentum density vectorfields $P(W; X_1, X_2, X_3)^\alpha$, see (32), with the vectorfields K, L in place of X_1, X_2, X_3.

[2] The $O_p(\delta^l |u_0|^s)$ denote the product of $\delta^l |u_0|^s$ with a non-negative and non-decreasing, continuous function of C^p-norm of ψ_0 on $[0,1] \times S^2$.

The commutation vectorfields are L, S and O_i with $i = 1, 2, 3$, where the latter are the rotation fields and S is defined by

$$S = u\underline{L} + \underline{u}L \ .$$

The modified Lie derivatives $\hat{\mathcal{L}}_Y$ of the curvature are taken with respect to the commutation vectorfields, thus Y replaced by $L, S, O_i : i = 1, 2, 3$. In particular there are first order and second order modified Lie derivatives of the spacetime curvature.

Next, one defines the total second order energy-momentum densities $P_2^{(n)}$ for $n = 0, 1, 2, 3$ as the sum of

$$\delta^{2l} \ P^{(n)}(W)$$

over all the Weyl fields in a specific way, where l denotes an index according to the number of $\hat{\mathcal{L}}_L$ operators applied to R. Then specific energies $E_2^{(n)}(u)$ are defined via integrals on C_u and fluxes $F_2^{(n)}(\underline{u})$ via integrals on the $\underline{C}_{\underline{u}}$ of the 3-forms dual to the $P_2^{(n)}$. From these, one defines the following quantities

$$\begin{aligned}
\mathcal{E}_2^{(n)} &= \sup_u \left(\delta^{2q_n} E_2^{(n)}(u)\right) \ , \quad n = 0, 1, 2, 3, && (36) \\
\mathcal{F}_2^{(3)} &= \sup_{\underline{u}} \left(\delta^{2q_3} F_2^{(3)}(\underline{u})\right) && (37)
\end{aligned}$$

with exponents $q_n : n = 0, 1, 2, 3$ given as follows

$$q_0 = 1 \ , \ q_1 = 0 \ , \ q_2 = -\frac{1}{2} \ , \ q_3 = -\frac{3}{2} \ .$$

One of the main goals then is to bound the quantities in (36), (37) in terms of the initial data.

We recall form above that the final piece in the proof of the existence theorem is to estimate all the other geometric quantities in a bootstrap argument under corresponding assumptions on the curvature. In particular, the behavior of the shear $\hat{\chi}$ will be crucial in view of (19) and theorems 8 respectively 9. In general, one may have the first idea to just integrate propagation equations for the connection coefficients along the generators of C_u and $\underline{C}_{\underline{u}}$, but in such a procedure a derivative would be "lost" because the connection coefficients are estimated at the same level as the curvature components. One has to make use of elliptic estimates for the connection coefficients on the $S_{\underline{u},u}$ and couple these with propagation equations. We already referred to this above, as Christodoulou and Klainerman developed this method in [27] to prove the global nonlinear stability of Minkowski spacetime.

These are the main ideas [24] of Christodoulou's proof of existence. Beyond these, more intricate challenges had to be overcome and the proof bears many conceptual and technical novelties that we do not address here. However, the main achievements summarized in this section allow us to understand the formation of closed trapped surfaces in theorems 8 and 9. This will be discussed in the next section.

4.5 Formation of Closed Trapped Surfaces

The closed trapped surface formation theorems 8 and 9 are simpler to prove than the existence theorem 10. From the proof of the latter, not only the spacetime is constructed that is required in theorems 8 respectively 9, but also the structure of the curvature and connection coefficients has been revealed. In particular the expansion $tr\underline{\chi}$ and the shear $\hat{\chi}$ are estimated.

From the existence result it follows that on M'_{-1} it is

$$|tr\underline{\chi} + \frac{2}{|u|}| \leq O(\delta|u|^{-2}) \quad . \tag{38}$$

Thus if δ is sufficiently small, then $tr\underline{\chi}$ is negative everywhere on M'_{-1}. Therefore, a surface $S_{\underline{u},u}$ in M'_{-1} is a indeed a *trapped* sphere if and only if everywhere on $S_{\underline{u},u}$ it is $tr\chi < 0$.

One also finds that on M'_{-1} it is

$$|\hat{\chi}|^2 \leq O(\delta^{-1}|u|^{-2}) \quad . \tag{39}$$

In the proof of the formation of closed trapped surfaces it is then shown that $tr\chi < 0$ which is deduced from the behavior of $\hat{\chi}$ that is from e of (19).

This concludes our discussion of the pioneering result [24] establishing the formation of closed trapped surfaces for the Einstein equations.

4.6 Generalization

Christodoulou's work [24] was generalized by Klainerman and Rodnianski in [51], [52]. Further extensions were given also in [48] by Klainerman, Luk and Rodnianski.

Klainerman and Rodnianski in [51], [52] relax the initial assumptions and therefore the short pulse hierarchy for closed trapped surfaces to form in the evolution of that data. Their proof mainly follows the lines of [24] by Christodoulou, but it introduces new solutions to new problems due to the more general situation.

The proof by Klainerman and Rodnianski on the one hand induces more challenges whereas on the other hand it simplifies certain aspects. A major simplification comes from the fact that the proof is established at one lower order of differentiability. However, as less is assumed from the beginning, less control is gained on the solutions than in the pioneering result.

Klainerman, Luk, and Rodnianski, using the original existence theorem, have derived a result [48] which extends considerably the original trapped surface formation theorem, as it does not require a lower bound on the incoming energy in all directions. In particular, they show the following: Consider the outer boundary $\underline{C}_\delta$ of the existence domain M_{-1}. Look at some neighborhood in S^2 of some direction and assume that the incoming energy in the directions corresponding to this neighborhood is sufficiently large depending on the angular size of this very neighborhood. Then $\underline{C}_\delta$ contains a trapped surface. It is interesting to note that even though none of the sections $S_{\delta,u}$ may be trapped, it is shown that there is another section of $\underline{C}_\delta$ that is trapped. Namely, this is a surface represented as a graph $u = G(\theta)$ over S^2. In fact, this surface attains large negative values of u in the directions corresponding to the neighborhood of large incoming energy, however, in the antipodal directions it comes near $S_{\delta,-1+\delta}$, the future boundary of $\underline{C}_\delta$ in M_{-1}.

Recently, P. Le [57] greatly simplified and clarified the latter work by looking at the intersection of a hyperplane with a lightcone in Minkowski spacetime. In view of the extrinsic geometry of the intersection, Le shows that in the case of a null hyperplane intersecting the cone, we have a non-compact marginally trapped surface. For this situation, he gives a geometric interpretation of the Green's function of the Laplacian on the standard sphere.

See also proof by Reiterer and Trubowitz [74]. Further, see Yu [91], Li and Yu [60]. Miao and Yu [63] applied the short pulse method successfully to study shock formation in quasilinear wave equations. Christodoulou's work [24] and his short pulse method have sparked a wealth of activity in GR and related fields. For a more complete discussion of these, see the latter references.

4.7 Incompleteness Theorem Revisited

As concluding remarks about black hole formation, we address again the incompleteness theorem in connection with the formation of closed trapped surfaces. Consider therefore the incompleteness theorem 1, respectively 2.

From this it can be shown in a straightforward manner that indeed the spacetime, which is the maximal development of "appropriate" initial data, contains a black hole region.

Christodoulou constructed explicit initial data in [24] in chapter 17 on page 579. For an appropriate choice of initial data which is also asymptotically flat (which is made precise in terms of decay of the data) there exists a solution spacetime for at least a finite value of u such that future null infinity $\mathcal{I}^+$ can be defined. Indeed, one can then prove the following corollary.

Corollary 1 *Consider complete initial data on C_0 as in theorem 8. Let (M, g) be the maximal development of this data. Then*

$$S_{\delta,-1-\delta} \cap J^-(I^+) = \emptyset$$

In particular, the spacetime M contains a black hole region B.

This corollary is obtained via the following road: Consider Christodoulou's initial data mentioned above together with his theorem 8 and Penrose's incompleteness theorem 1 (respectively 2). From this, the next statement follows directly.

Corollary 2 *Consider complete initial data on C_0 as in theorem 8. Let (M, g) be the maximal development of this data. Then (M, g) is future causally geodesically incomplete.*

We can summarize simply that the presence of a closed trapped surface in a spacetime (M, g), which solves the Einstein vacuum equations, implies the existence of a black hole. By a simple contradiction argument we can prove that, if such a (M, g) contains a closed trapped surface S, then $S \cap J^-(I^+) = \emptyset$, which is the content of corollary 1. In particular, it says that S cannot lie in the domain of outer communications $J^-(I^+)$, but M must have a black hole region that must contain S.

See the following references on these topics: [46], [90], [26], as well as [27], [25], [49].

5 Stability of Black Holes

If we think of the simplest black hole solutions of the EV equations (1), namely Schwarzschild or Kerr spacetimes, then we may ask: What happens, if we perturb such a specific solution? Do we expect it to settle down to (another) solution of that type? Maybe yes, if we start with initial data that is sufficiently close to Schwarzschild or Kerr? It turns out that this is an open problem of very active research in mathematical GR with goal of proving the nonlinear stability of the Kerr family. In fact, one would like to prove a conjecture of the following form: (Mass is denoted by M and angular momentum by a.)

Conjecture 1 *Let $(H, \bar{g}, k)$ be a vacuum initial data set sufficiently close to two-ended Kerr data for some subextremal parameters $0 \leq |a_i| \leq M_i$. Then the resulting vacuum spacetime (M, g) has a complete future null infinity $\mathcal{I}^+$ such that the metric restricted to $J^-(\mathcal{I}^+)$ remains close to for all time and moreover asymptotically settles down to a nearby Kerr solution in a uniform way with quantitative decay rates.*

An important fact for the Einstein equations is, that the problem of completeness and asymptotic stability in the above sense are coupled. Therefore, any progress towards understanding those problems has to come with a quantitative description of decay rates of the solution.

As Kerr stability is a huge field with many contributors, instead of citing the extremely long list of partial achievements, we refer to the following survey articles for a precise account of the history and references including the latest works; Dafermos [28], Tataru [85], Andersson, Bäckdahl, Blue [2].

One can formulate the following three statements of "stability" in this context: linear mode stability, linear stability and nonlinear stability (see conjecture). Whiting proved linear mode-stability based on work by Teukolsky. (In Schwarzschild case, see Regge-Wheeler, Zerilli.) A related problem, that one wants to solve before attacking the main case above, is the study of the wave equation on a fixed black hole background. For this case, versions of linear stability on Schwarzschild were proven by Wald, Kay, Friedman.

In recent years, the main focus was on studying the wave equation on Kerr background (see works by Andersson, Blue, Dafermos, Dyatlov, Finster, Ionescu, Kamran, Klainerman, Rodnianski, Smoller, Sterbenz, Tataru, Tohaneanu, Yau and many more). We may summarize the findings as: Solutions to the wave equation $\Box_g \Psi = 0$ on subextremal Kerr $|a| < M$ remain bounded in the exterior and decay inverse-polynomially to zero. An interesting linear instability arises (see Aretakis, Lucietti-Reall) in the extremal case, which is not captured by "mode stability".

It was recently established that solutions to the linearisation of the EV equations around a Schwarzschild metric for regular initial data remain globally bounded on the black hole exterior, decay to a linearised Kerr metric. See recent paper by Dafermos, Holzegel and Rodnianski, and recent paper by Finster and Smoller.

The only global nonlinear stability result (theorem 7) proven so far is the one on the global nonlinear stability of Minkowski spacetime of [27] by Christodoulou and Klainerman. See also generalizations and other related works cited above.

On the road towards proving the fully nonlinear result of the above conjecture, there are expected to be many beautiful insights into the structures of the Einstein equations.

6 Cosmic Censorship

One of the great open problems of GR is the so-called "weak cosmic censorship" conjecture which is the following:

Conjecture 2 (Weak Cosmic Censorship) *For generic asymptotically flat vacuum initial data, the resulting vacuum spacetime has a complete future null infinity $\mathcal{I}^+$.*

There is of course room to make clear what "generic" data should look like.

The Einstein equations allow also other types of singularities, namely the so-called naked singularities, which are not surrounded by an event horizon but can be seen from infinity. Cosmic censorship conjectures that the latter do not form during a gravitational collapse. Christodoulou investigated this in a series of papers in the 1980s and 1990s, where he showed for certain classes of initial data (studying a scalar-field model) that such naked singularities may occur, but they are unstable. See his work [24] for a summary and the references.

For a nice explanation of the above conjecture and also the "strong cosmic censorship" statement, see the introduction of [24].

There is still a long way to go in order to understand trapped surface formation for the general Einstein equations (2), when the right hand side is not equal to zero. A huge step happened with Christodoulou's result [24] constituting a major breakthrough establishing the picture outlined above for the Einstein vacuum equations (1).

Beyond the topics addressed in this article, mathematical GR bears many more exciting challenges for geometric analysis and other mathematical fields to be investigated in the future.

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HODGE THEORY IN COMBINATORICS

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Abstract. If G is a finite graph, a *proper coloring* of G is a way to color the vertices of the graph using n colors so that no two vertices connected by an edge have the same color. (The celebrated four-color theorem asserts that if G is *planar* then there is at least one proper coloring of G with 4 colors.) Hassler Whitney proved in 1932 that the number of proper colorings of G with n colors is a polynomial in n, called the *chromatic polynomial* of G. Read conjectured in 1968 that for any graph G, the sequence of absolute values of coefficients of the chromatic polynomial is unimodal: it goes up, hits a peak, and then goes down. Read's conjecture was proved by June Huh in a 2012 paper [12] making heavy use of methods from algebraic geometry. Huh's result was subsequently refined and generalized by Huh and Katz [13], again using substantial doses of algebraic geometry. Both papers in fact establish *log-concavity* of the coefficients, which is stronger than unimodality.

The breakthroughs of Huh and Huh–Katz left open the more general Rota–Welsh conjecture where graphs are generalized to (not necessarily representable) matroids and the chromatic polynomial of a graph is replaced by the characteristic polynomial of a matroid. The Huh and Huh–Katz techniques are not applicable in this level of generality, since there is no underlying algebraic geometry to which to relate the problem. But in 2015 Adiprasito, Huh, and Katz [1] announced a proof of the Rota–Welsh conjecture based on a novel approach motivated by but not making use of any results from algebraic geometry. The authors first prove that the Rota–Welsh conjecture would follow from combinatorial analogues of the Hard Lefschetz Theorem and Hodge-Riemann relations in algebraic geometry. They then implement an elaborate inductive procedure to prove the combinatorial Hard Lefschetz Theorem and Hodge-Riemann relations using purely combinatorial arguments.

We will survey these developments.

1. Unimodality and Log-Concavity

A sequence $a_0, \ldots, a_d$ of real numbers is called *unimodal* if there is an index i such that

$$a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_d.$$

There are numerous naturally-occurring unimodal sequences in algebra, combinatorics, and geometry. For example:

Example 1.1. (Binomial coefficients) The sequence of binomial coefficients $\binom{n}{k}$ for n fixed and $k = 0, \ldots, n$ (the n^{th} row of Pascal's triangle) is unimodal.

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The sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ has a property which is in fact stronger than unimodality: it is *log-concave*, meaning that $a_i^2 \geq a_{i-1}a_{i+1}$ for all i. Indeed,

$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(k+1)(n-k+1)}{k(n-k)} > 1.$$

It is a simple exercise to prove that a log-concave sequence of positive numbers is unimodal.

Some less trivial, but still classical and elementary, examples of log-concave (and hence unimodal) sequences are the Stirling numbers of the first and second kind.

Example 1.2. (Stirling numbers) The *Stirling numbers of the first kind*, denoted $s(n,k)$, are the coefficients which appear when one writes falling factorials $(x)_n = x(x-1)\cdots(x-n+1)$ as polynomials in x:

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k.$$

This sequence of integers alternates in sign. The *signless Stirling numbers of the first kind* $s^+(n,k) = |s(n,k)| = (-1)^{n-k}s(n,k)$ enumerate the number of permutations of n elements having exactly k disjoint cycles.

The *Stirling numbers of the second kind*, denoted $S(n,k)$, invert the Stirling numbers of the first kind in the sense that

$$\sum_{k=0}^{n} S(n,k)(x)_k = x^n.$$

Their combinatorial interpretation is that $S(n,k)$ counts the number of ways to partition an n element set into k non-empty subsets.

For fixed n (with k varying from 0 to n), both $s^+(n,k)$ and $S(n,k)$ are log-concave and hence unimodal.

Another example, proved much more recently through a decidedly less elementary proof, concerns the sequence of coefficients of the chromatic polynomial of a graph. This example will be the main focus of our paper.

Example 1.3. (Coefficients of the chromatic polynomial) Let G be a connected finite graph. In 1932, Hassler Whitney defined $\chi_G(t)$ to be the number of proper colorings of G using t colors (i.e., the number of functions $f : V(G) \to \{1, \ldots, t\}$ such that $f(v) \neq f(w)$ whenever v and w are adjacent in G), and proved that $\chi_G(t)$ is a *polynomial* in t, called the **chromatic polynomial** of G.

For example, if $G = T$ is a tree on n vertices then the chromatic polynomial of G is

$$\chi_T(t) = t(t-1)^{n-1} = \sum_{k=1}^{n} (-1)^{n-k}\binom{n-1}{k-1}t^k.$$

If $G = K_n$ is the complete graph on n vertices, then

$$\chi_{K_n}(t) = (t-1)(t-2)\cdots(t-n+1) = \sum_{k=0}^{n} s(n,k)t^k.$$

And if G is the Petersen graph, depicted in Figure 1, then

$$\chi_G(t) = t^{10} - 15t^9 + 105t^8 - 455t^7 + 1353t^6 - 2861t^5 + 4275t^4 - 4305t^3 + 2606t^2 - 704t.$$

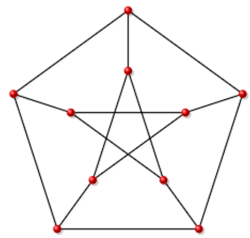

FIGURE 1. The Petersen graph

Ronald Reed conjectured in 1968 that for any graph G the (absolute values of the) coefficients of $\chi_G(t)$ form a unimodal sequence, and a few years letter Stuart Hoggar conjectured that the coefficients in fact form a log-concave sequence. Both conjectures were proved only relatively recently by June Huh [12].

Another interesting and relevant example concerns linearly independent sets of vectors:

Example 1.4. Let k be a field, let V be a vector space over k, and let A be a finite subset of V. Dominic Welsh conjectured that $f_i(A)$ is a log-concave sequence, where $f_i(A)$ is the number of linearly independent subsets of A of size i. For example, if $k = \mathbb{F}_2$ is the field of 2 elements, $V = \mathbb{F}_2^3$, and $A = V \backslash \{0\}$, then

$$f_0(A) = 1, f_1(A) = 7, f_2(A) = 21, f_3(A) = 28.$$

This conjecture is a consequence of the recent work of Huh–Katz [13] (cf. [16]).

Finally, we mention an example of an apparently much different nature coming from algebraic geometry:

Example 1.5. (Hard Lefschetz Theorem) Let X be an irreducible smooth projective algebraic variety of dimension n over the field $\mathbb{C}$ of complex numbers, and let $\beta_i = \dim H^i(X, \mathbb{C})$ be the i^{th} *Betti number* of X. (Here $H^*(X, \mathbb{C})$ denotes the singular cohomology groups of X.) Then the two sequences $\beta_0, \beta_2, \ldots, \beta_{2n}$ and $\beta_1, \beta_3, \ldots, \beta_{2n-1}$ are symmetric and unimodal. Moreover, this remains true if we replace the hypothesis that X is smooth by the weaker hypothesis that X has only finite quotient singularities, meaning that X looks locally (in the analytic topology) like the quotient of $\mathbb{C}^n$ by a finite group of linear transformations.

The symmetry of the β_i's is a classical result in topology known as *Poincaré duality*. And one has the following important strengthening (given symmetry) of unimodality: there is an element $\omega \in H^2(X, \mathbb{C})$ such that for $0 \leq i \leq n$, multiplication by ω^{n-i} defines an isomorphism from $H^i(X, \mathbb{C})$ to $H^{2n-i}(X, \mathbb{C})$. This result is called the *Hard Lefschetz Theorem*. In the smooth case, it is due to Hodge; for varieties with finite quotient singularities, it is due to Saito and uses the theory of perverse sheaves.

For varieties X with arbitrary singularities, the Hard Lefschetz Theorem still holds if one replaces singular cohomology by the *intersection cohomology* of Goresky and MacPherson (cf. [5]).

Surprisingly, all five of the above examples are in fact related. We have already seen that Example 1.1, as well as Example 1.2 in the case of Stirling numbers of the first kind, are special cases of Example 1.3. We will see in the next section that Examples 1.3 and 1.4 both follow from a more general result concerning **matroids**. And the proof of this theorem about matroids will involve, as one of its key ingredients, a combinatorial analogue of the Hard Lefschetz Theorem (as well as the Hodge-Riemann relations, about which we will say more later).

2. Matroids

Our primary references for this section are [19] and [21].

2.1. **Independence axioms.** Matroids were introduced by Hassler Whitney as a combinatorial abstraction of the notion of linear independence of vectors. There are many different ("cryptomorphic") ways to present the axioms for matroids, all of which turn out to be non-obviously equivalent to one another. For example, instead of using linear independence one can also define matroids by abstracting the notion of *span*. We will give a brief utilitarian introduction to matroids, starting with the independence axioms.

Definition 2.1. (Independence Axioms) A **matroid** M is a finite set E together with a collection $\mathcal{I}$ of subsets of E, called the *independent sets* of the matroid, such that:

- (I1) The empty set is independent.
- (I2) Every subset of an independent set is independent.
- (I3) If I, J are independent sets with $|I| < |J|$, then there exists $y \in J \backslash I$ such that $I \cup \{y\}$ is independent.

2.2. **Examples.**

Example 2.2. (Linear matroids) Let V be a vector space over a field k, and let E be a finite subset of V. Define $\mathcal{I}$ to be the collection of linearly independent subsets of E. Then $\mathcal{I}$ satisfies (I1)-(I3) and therefore defines a matroid. Matroids of this form are called **representable** over k. If we write the elements of E as the columns of an $m \times n$ matrix A (with respect to some ordered basis of V), then a subset of E is independent iff the corresponding columns of A are linearly independent over k. We denote this matroid by $M_k(A)$. Up to (the obvious notion of) isomorphism, this matroid depends only on the row space of A. We may thus think of a linear matroid on E as a vector subspace V of k^E.

By a recent theorem of Peter Nelson [18], asymptotically 100% of all matroids are not representable over any field.

Example 2.3. (Graphic matroids) Let G be a connected finite graph, let E be the set of edges of G, and let $\mathcal{I}$ be the collection of all subsets of E which do not contain a cycle. Then $\mathcal{I}$ satisfies (I1)-(I3) and hence defines a matroid $M(G)$. The matroid $M(G)$ is **regular**, meaning that it is representable over every field k. By a theorem of Whitney, if G is 3-connected (meaning that G remains connected after removing any two vertices) then $M(G)$ determines the isomorphism class of G.

Example 2.4. (Uniform matroids) Let $E = \{1, \ldots, m\}$ and let r be a positive integer. The **uniform matroid** $U_{r,m}$ is the matroid on E whose independent sets are the subsets of E of cardinality at most r. For each r, m there exists $N = N(r, m)$ such that $U_{r,m}$ is representable over every field having at least $N(r, m)$ elements.

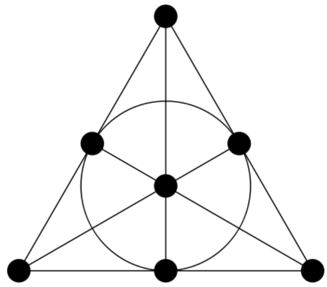

FIGURE 2. The Fano matroid

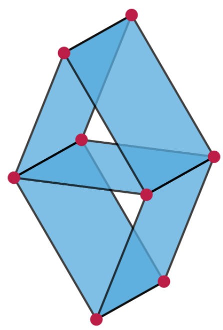

FIGURE 3. The Vamos matroid

Example 2.5. (Fano matroid) Let $E = \mathbb{P}^2(\mathbf{F}_2)$ be the projective plane over the 2-element field; the seven elements of E can be identified with the dots in Figure 2.

Define $\mathcal{I}$ to be the collection of subsets of E of size at most 3 which are not one of the 7 lines in $\mathbb{P}^2(\mathbf{F}_2)$ (depicted as six straight lines and a circle in Figure 2). Then $\mathcal{I}$ satisfies (I1)-(I3) and determines a matroid called the *Fano matroid*. This matroid is representable over $\mathbf{F}_2$ but not over any field of characteristic different from 2. In particular, the Fano matroid is not graphic.

Example 2.6. (Vamos matroid) Let E be the 8 vertices of the cuboid shown in Figure 3. Define $\mathcal{I}$ to be the collection of subsets of E of size at most 4 which are not one of the five square faces in the picture. Then $\mathcal{I}$ satisfies (I1)-(I3) and determines a matroid called the *Vamos matroid* which is not representable over any field.

2.3. **Circuits, bases, and rank functions.** A subset of E which is not independent is called dependent. A minimal dependent set is called a **circuit**, and a maximal independent set is called a **basis**. As in linear algebra, all bases of M have the same cardinality; this number is called the **rank** of the matroid M, and is denoted $r(M)$. More generally, if A is a subset of E, we define the **rank** of A, denoted $r_M(A)$ or just $r(A)$, to be the maximal size of an independent subset of A.

One can give cryptomorphic axiomatizations of matroids in terms of circuits, bases, and rank functions. For the sake of brevity we refer the interested reader to [19].

2.4. **Duality.** If $M = (E, \mathcal{I})$ is a matroid, let $\mathcal{I}^*$ be the collection of subsets $A \subseteq E$ such that $E \backslash A$ contains a basis B for M. It turns out that $\mathcal{I}^*$ satisfies axioms (I1),(I2), and (I3) and thus $M^* = (E, \mathcal{I}^*)$ is a matroid, called the **dual matroid** of M.

If $M = M_k(A)$ is the linear matroid associated to a matrix A, then M^* is the linear matroid associated to the *transpose* of A. (In terms of subspaces, if M corresponds to $V \subset k^m$ then M^* corresponds to the orthogonal complement of V.)

If $M = M(G)$ is the matroid associated to a planar graph G, then M^* is the matroid associated to the *planar dual* of G. A theorem of Whitney asserts, conversely, that if G is a connected graph for which the dual matroid $M(G)^*$ is graphic, then G is planar.

2.5. **Deletion and Contraction.** Given a matroid M on E and $e \in E$, we write $M \backslash e$ for the matroid on $E \backslash \{e\}$ such that I is independent in $M \backslash e$ if and only if $I = J|_{E \backslash \{e\}}$ with J independent in M and $e \notin J$.

We write M/e for the matroid on $E \backslash \{e\}$ such that I is independent in M/e if and only if $I = J \backslash \{e\}$ with J independent in M and $e \in J$.

We call these operations on matroids **deletion** and **contraction**, respectively. Deletion and contraction are dual operations, in the sense that $(M \backslash e)^* = M^*/e$ and $(M/e)^* = M^* \backslash e$.

If M is a graphic matroid, deletion and contraction correspond to the usual notions in graph theory.

2.6. **Spans.** We defined matroids in terms of independent sets, which abstract the notion of linear independence. We now focus on a different way to define / characterize matroids in terms of *closure operators*, which abstract the notion of *span* in linear algebra.

Let 2^E denote the power set of E.

Definition 2.7. (Span Axioms) A **matroid** M is a finite set E together with a function $\mathrm{cl} : 2^E \to 2^E$ such that for all $X, Y \subseteq E$ and $x, y \in E$:

(S1) $X \subseteq \mathrm{cl}(X)$.
(S2) If $Y \subseteq X$ then $\mathrm{cl}(Y) \subseteq \mathrm{cl}(X)$.
(S3) $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.
(S4) If $y \in \mathrm{cl}(X \cup \{x\})$ but $y \notin \mathrm{cl}(X)$, then $x \in \mathrm{cl}(X \cup \{y\})$.

For example, if M is a linear matroid as in Example 2.2 then $\mathrm{cl}(X)$ is just the span of X in V.

The *exchange axiom* (S4) captures our intuition of a "geometry" as a collection of incidence relations

$$\{\text{point}\} \subset \{\text{line}\} \subset \{\text{plane}\} \subset \cdots$$

For example, if L is a line in $(r+1)$-dimensional projective space $\mathbb{P}_k^{r+1}$ over a field k and $p, q \in \mathbb{P}_k^{r+1} \backslash L$, then q lies in the span of $L \cup \{p\} \Leftrightarrow p$ lies in the span of $L \cup \{q\} \Leftrightarrow p, q, L$ are *coplanar*.

The relation between Definitions 2.1 and 2.7 is simple to describe: given a matroid in the sense of Definition 2.1, we define $\mathrm{cl}(X)$ to be the union of X and all

$x \in E$ such that $X \cup \{x\}$ is dependent (i.e., does not belong to $\mathcal{I}$). Conversely, given a matroid in the sense of Definition 2.7, we define a subset I of E to be independent if and only if $x \in I$ implies $x \notin \mathrm{cl}(I \backslash \{x\})$.

A subset X of E is said to **span** M if $\mathrm{cl}(X) = E$. As in the familiar case of linear algebra, one can show in general that X is a basis (i.e., a maximal independent set) if and only if X is independent and spans E.

2.7. **Flats.** A subset X of E is called a **flat** (or a *closed subset*) if $X = \mathrm{cl}(X)$.

Example 2.8. (Linear matroids) Let V be a vector space and let E be a finite subset of V. A subset F of E is a flat of the corresponding linear matroid if and only if there is no vector in $E \backslash F$ contained in the linear span of F.

Alternatively, let $M = M_k(A)$ be represented by an $r \times m$ matrix A of rank r with entries in k, and let $V \subseteq k^m$ be the row space of A. Let $E = \{1, \ldots, m\}$, and for $I \subseteq E$ let L_I be the "coordinate flat"

$$L_I = \{x = (x_1, \ldots, x_m) \in k^m \ : \ x_i = 0 \text{ for } i \in I\}.$$

Then for $I \subseteq E$ we have $r_M(I) = \mathrm{codim}(V \cap L_I)$, and I is a flat of M if and only if $V \cap L_J \subsetneq V \cap L_I$ for all $J \supsetneq I$. In particular, $V \cap L_I = V \cap L_F$, where F is the smallest flat of M containing I.

Example 2.9. (Graphic matroids) Let G be a connected finite graph, and let $M(G)$ be the associated matroid. Then a subset F of E is a flat if and only if there is no edge in $E \backslash F$ whose endpoints are connected by a path in F.

Example 2.10. (Fano matroid) In the Fano matroid, the flats are $\emptyset, E$, and each of the 7 points and 7 lines in Figure 2.

Every maximal chain of flats of a matroid M has the same length, which coincides with the rank of M.

One can give a cryptomorphic axiomatization of matroids in terms of flats. To state it, we say that a flat F' **covers** a flat F if $F \subsetneq F'$ and there are no intermediate flats between F and F'.

Definition 2.11. (Flat Axioms) A **matroid** M is a finite set E together with a collection of subsets of E, called **flats**, such that:[1]

(F1) E is a flat.
(F2) The intersection of two flats is a flat.
(F3) If F is a flat and $\{F_1, F_2, \ldots, F_k\}$ is the set of flats that cover F, then $\{F_1 \backslash F, F_2 \backslash F, \ldots, F_k \backslash F\}$ partitions $E \backslash F$.

We have already seen how to define flats in terms of a closure operator. To go the other way, one defines the closure of a set X to be the intersection of all flats containing X.

2.8. **Simple matroids.** A matroid M is called **simple** if every dependent set has size at least 3. Equivalently, a matroid is simple if and only if it has no:

- **Loops** (elements $e \in \mathrm{cl}(\emptyset)$); or
- **Parallel elements** (elements e, e' with $e' \in \mathrm{cl}(e)$).

[1]For the geometric intuition behind axiom (F3), note that given a line in $\mathbb{R}^3$, the plans which contain this line partition the remainder of $\mathbb{R}^3$.

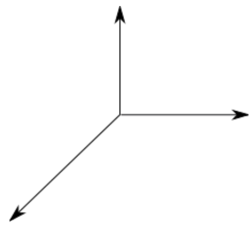

FIGURE 4. The Bergman fan of $U_{2,3}$

Every matroid M has a canonical **simplification** $\hat{M}$ obtained by removing all loops and identifying parallel elements (with the obvious resulting notions of independence, closure, etc.). A simple matroid is also called a **combinatorial geometry**.

For future reference, we define a **coloop** of a matroid M to be a loop of M^*. Equivalently, a loop is an element $e \in E$ which does not belong to any basis of M, and a coloop is an element $e \in E$ which belongs to every basis of M.

2.9. **The Bergman fan of a matroid.** Let $E = \{0, 1, \ldots, n\}$ and let M be a matroid on E. The Bergman fan of M is a certain collection of cones in the n-dimensional Euclidean space $N_{\mathbb{R}} = \mathbb{R}^E/\mathbb{R}(1, 1, \ldots, 1)$ which carries the same combinatorial information as M. Bergman fans show up naturally in the context of tropical geometry, where they are also known as *tropical linear spaces*.

For $S \subseteq E$, let $e_S = \sum_{i \in S} e_i \in N_{\mathbb{R}}$, where e_i is the basis vector of $\mathbb{R}^E$ corresponding to i. Note that $e_E = 0$ by the definition of $N_{\mathbb{R}}$. Let $F_\bullet = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\}$ be a k-step flag of non-empty proper flats of M. We define the corresponding cone $\sigma_{F_\bullet} \subseteq N_{\mathbb{R}}$ to be the nonnegative span of the e_{F_i} for $i = 1, \ldots, k$.

Definition 2.12. The **Bergman fan** Σ_M of M is the collection of cones $\sigma_{F_\bullet}$ as $F_\bullet$ ranges over all flags of non-empty proper flats of M.

Example 2.13. The Bergman fan of the uniform matroid $U_{2,3}$ has a zero-dimensional cone given by the origin in $\mathbb{R}^2$ and three 1-dimensional cones given by rays from the origin in the directions of $\bar{e}_1 = (1, 0), \bar{e}_2 = (0, 1)$, and $\bar{e}_3 = -(\bar{e}_1 + \bar{e}_2) = (-1, -1)$. This is the well-known "tropical line" in $\mathbb{R}^2$ with vertex at the origin, see Figure 4.

Example 2.14. Let $U = U_{n+1,n+1}$ be the rank $n + 1$ uniform matroid on $E = \{0, 1, \ldots, n\}$. Every subset of E is a flat, so the top-dimensional cones of Σ_U are the nonnegative spans of

$$\{e_{i_0}, e_{i_0} + e_{i_1}, \ldots, e_{i_0} + e_{i_1} + \cdots + e_{i_{n-1}}\}$$

for every permutation $i_0, \ldots, i_n$ of $0, \ldots, n$. The fan Σ_U is the normal fan to the **permutohedron** P_n. (The permutohedron is by definition the convex hull of $(i_0, \ldots, i_n)$ over all permuations $i_0, \ldots, i_n$ of $0, \ldots, n$, viewed as a polytope in $N_{\mathbb{R}}$, see Figure 5.) The fan Σ_U plays a central and recurring role in [1].

The isomorphism class of matroid M determines and is determined by its Bergman fan.[2]

[2]One can in fact give a cryptomorphic characterization of matroids via their Bergman fans, using the flat axioms (F1)-(F3): a rational polyhedral fan Σ in $N_{\mathbb{R}}$ is the Bergman fan of a matroid on E if and only if it is *balanced* and has *degree one* as a tropical cycle [10].

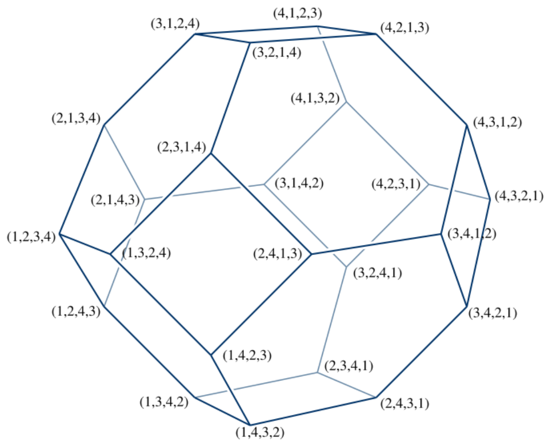

Figure 5. The 3-dimensional permutohedron. Note that here we take $E = \{1, 2, 3, 4\}$, instead of $\{0, 1, 2, 3\}$ as in Example 2.14, but this does not matter in $N_{\mathbb{R}} = \mathbb{R}^E/\mathbb{R}(1, 1, \ldots, 1)$.

3. Geometric lattices and the characteristic polynomial

In this section we define the characteristic polynomial of a matroid M in terms of the lattice of flats of M. Our primary references are [21] and [22].

3.1. **Geometric lattices.** The set $\mathcal{L}(M)$ of flats of a matroid M together with the inclusion relation forms a **lattice**, i.e., a partially ordered set in which every two elements x, y have both a **meet** (greatest lower bound) $x \wedge y$ and a **join** (least upper bound) $x \vee y$. Indeed, if X and Y are flats then we can define $X \wedge Y$ as the intersection of X and Y and $X \vee Y$ as the closure of the union of X and Y.

Example 3.1. Flats of the uniform matroid $U_{n+1,n+1}$ can be identified with subsets of $\{0, 1, \ldots, n\}$, and with this identification the lattice of flats of $U_{n+1,n+1}$ is the **Boolean lattice** B_{n+1} consisting of subsets of $\{0, 1, \ldots, n\}$ partially ordered by inclusion.

Example 3.2. Flats of the complete graph K_n can be identified with partitions of $\{1, \ldots, n\}$, and with this identification the lattice of flats of the graphic matroid $M(K_n)$ is isomorphic to the **partition lattice** Π_n consisting of partitions of $\{1, \ldots, n\}$ partially ordered by reverse refinement.

If L is a lattice and $x, y \in L$, we say that y **covers** x if $x < y$ and whenever $x \leq z \leq y$ we have either $z = x$ or $z = y$. A finite lattice has a minimal element 0_L and a maximal element 1_L. An **atom** is an element which covers 0_L.

The lattice of flats $L = \mathcal{L}(M)$ has the following properties:

(L1) L is **semimodular**, i.e., if $x, y \in L$ both cover $x \wedge y$ then $x \vee y$ covers both x and y.
(L2) L is **atomic**, i.e., every $x \in L$ is a join of atoms.

A lattice satisfying (L1) and (L2) is called a **geometric lattice**. By a theorem of Birkhoff, every geometric lattice is of the form $\mathcal{L}(M)$ for some matroid M. However, the matroid M is not unique, because if $\hat{M}$ is the simplification of M then $\mathcal{L}(M) = \mathcal{L}(\hat{M})$. However, Birkhoff proves that this is the only ambiguity, i.e., the map $M \mapsto \mathcal{L}(M)$ gives a bijection between isomorphism classes of simple matroids and isomorphism classes of geometric lattices. Thus, at least up to simplification, (L1) and (L2) give another cryptomorphic characterization of matroids.

If F is a flat of a matroid M, the maximal length ℓ of a chain $F_0 \subset F_1 \subset \cdots \subset F_\ell = F$ of flats coincides with the rank $r_M(F)$ of F. This allows us to define the rank function on M, restricted to the set of flats, purely in terms of the lattice $\mathcal{L}(M)$. We write r_L for the corresponding function on an arbitrary geometric lattice L.

3.2. **The Möbius function of a poset.** There is a far-reaching combinatorial abstraction of the Inclusion-Exclusion Principle called the Möbius Inversion Formula which holds in an arbitrary finite poset P.

There is a unique function $\mu_P : P \times P \to \mathbb{Z}$, called the **Möbius function** of P, satisfying $\mu_P(x,x) = 1$, $\mu_P(x,y) = 0$ if $x \not\leq y$, and

$$\sum_{x \leq z \leq y} \mu_P(x,z) = 0.$$

if $x < y$. Note that $\mu_P(x,y) = -1$ if y covers x.

The **Möbius Inversion Formula** states that if f is a function from a finite poset P to an abelian group H, and if we define $g(y) = \sum_{x \leq y} f(x)$ for all $y \in P$, then

$$f(y) = \sum_{x \leq y} \mu_P(x,y) g(x).$$

If $P = L$ is a finite *lattice*, the Möbius function satisfies **Weisner's theorem**, which gives a "shortcut" for the recurrence defining μ: if $0_L \neq x \in L$ then

$$\sum_{y \in L \,:\, x \vee y = 1_L} \mu_L(0_L, y) = 0.$$

If L is moreover a *geometric lattice*, it is a theorem of Rota that the Möbius function of L is non-zero and alternates in sign. More precisely, if $x \leq y$ in L then

$$(-1)^{r_L(y) - r_L(x)} \mu_L(x,y) > 0.$$

3.3. **The characteristic polynomial.** The chromatic polynomial of a graph G satisfies the *deletion-contraction relation*:

$$\chi_G(t) = \chi_{G \backslash e}(t) - \chi_{G/e}(t).$$

(Indeed, the left-hand side counts the proper colorings of G, first term on the right-hand side counts the otherwise-proper colorings of G where the endpoints of e are allowed to have the same color, and the second term on the right-hand side counts the otherwise-proper colorings of G where the endpoints of e are required to have the same color.)

This formula is not only useful for calculating $\chi_G(t)$, it is also the simplest way to prove that $\chi_G(t)$ is a polynomial in t. In addition, this formula for $\chi_G(t)$ suggests an extension to arbitrary matroids. This can be made to work, but it is not obvious that this recursive procedure is always well-defined. So it is more convenient to proceed as follows.

First, note that the chromatic polynomial of a graph G is identically zero by definition if G has a loop edge. So we will define $\chi_M(t) = 0$ for any matroid with a loop. We may thus concentrate on loopless matroids. Note that a matroid M is loopless if and only if $\emptyset$ is a flat of M.

Definition 3.3. Let M be a loopless matroid with lattice of flats L. The **characteristic polynomial** of M is

$$\chi_M(t) = \sum_{F \in L} \mu_L(\emptyset, F) t^{r(M)-r(F)}. \tag{3.1}$$

In particular, if M is loopless then $\chi_M(t) = \chi_{\hat{M}}(t)$, where $\hat{M}$ denotes the simplification of M.

The motivation behind (3.1) may be unclear to the reader at this point. In the representable case, at least, there is a "motivic" interpretation of (3.1) which some will find illuminating; see §3.6.

There is also a (simpler-looking but less useful) expression for $\chi_M(t)$ in terms of a sum over *all* subsets of E, not just flats.

Proposition 3.4. *If M is any matroid,*

$$\chi_M(t) = \sum_{A \subseteq E} (-1)^{|A|} t^{r(M)-r(A)}. \tag{3.2}$$

If M_1, M_2 are matroids on E_1 and E_2, respectively, and $E_1 \cap E_2 = \emptyset$, we define the **direct sum** $M_1 \oplus M_2$ to be the matroid on $E_1 \cup E_2$ whose flats are all sets of the form $F_1 \cup F_2$ where F_i is a flat of M_i for $i = 1, 2$. The following result gives an important characterization of the characteristic polynomial.

Theorem 3.5. *Let M be a matroid.*

- (χ1) *If e is neither a loop nor a coloop of M, then $\chi_M(t) = \chi_{M \setminus e}(t) - \chi_{M/e}(t)$.*
- (χ2) *If $M = M_1 \oplus M_2$ then $\chi_M(t) = \chi_{M_1}(t)\chi_{M_2}(t)$.*
- (χ3) *If M contains a loop then $\chi_M(t) = 0$, and if M consists of a single coloop then $\chi_M(t) = t - 1$.*

Furthermore, the characteristic polynomial is the unique *function from matroids to integer polynomials satisfying* (χ1)-(χ3).

In particular, it follows from Theorem 3.5 that if G is a graph then the chromatic polynomial $\chi_G(t)$ of G satisfies $\chi_G(t) = t^{c(G)}\chi_{M(G)}(t)$, where $c(G)$ is the number of connected components of G. (The extra factor of t comes from the fact that the graph with two vertices and one edge has chromatic polynomial $t(t-1)$, whereas the corresponding matroid, which consists of a single coloop, has characteristic polynomial $t - 1$. Note that since no graph can be 0-colored, $\chi_G(0) = 0$ for every graph G and hence the chromatic polynomial is always divisible by t.)

The characteristic polynomial of M is monic of degree $r = r(M)$, so we can write

$$\chi_M(t) = w_0(M)t^r + w_1(M)t^{r-1} + \cdots + w_r(M)$$

with $w_0(M) = 1$ and $w_k(M) \in \mathbb{Z}$. By Rota's theorem, the coefficients of $\chi_M(t)$ alternate in sign, i.e.,

$$w_k(M)^+ := |w_k(M)| = (-1)^k w_k(M).$$

The numbers $w_k(M)$ (resp. $w_k^+(M)$) are called the **Whitney numbers of the first kind** (resp. **unsigned Whitney numbers of the first kind**) for M. The recent work of Adiprasito–Huh–Katz [1] establishes:

Theorem 3.6. *For any matroid M, the unsigned Whitney numbers of the first kind $w_k^+(M)$ form a log-concave sequence.*

Note that it is enough to prove the theorem for simple matroids, i.e., combinatorial geometries, since the characteristic polynomial of a loopless matroid equals that of its simplification.

Actually, Adiprasito, Huh, and Katz study the so-called **reduced characteristic polynomial** of M. If $|E| \geq 1$ then $\chi_M(1) = 0$ (e.g., if G is a graph with at least one edge then G has no proper one-coloring!). Thus we may write $\chi_M(t) = (t-1)\bar{\chi}_M(t)$ with $\bar{\chi}_M(t) \in \mathbb{Z}[t]$. The reduced characteristic polynomial $\bar{\chi}_M(t)$ is the "projective analogue" of $\chi_M(t)$ (cf. §3.6 below). It is an elementary fact that log-concavity of the (absolute values of the) coefficients of $\bar{\chi}_M(t)$ implies log-concavity for $\chi_M(t)$. So in order to prove Theorem 3.6 one can replace the $w_k^+(M)$ by their projective analogues $m_k(M)$.

3.4. **Tutte-Grothendieck invariants.** Our primary reference for this section and §3.6 is [15].

One can generalize the characteristic polynomial of a matroid by relaxing the condition that it vanishes on matroids containing loops.

The **Tutte polynomial** of a matroid M on E is the two-variable polynomial

$$T_M(x,y) = \sum_{A \subseteq E} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}.$$

By (3.2), we have $\chi_M(t) = (-1)^{r(M)} T_M(1-t, 0)$.

To put Theorem 3.5 into perspective, we define the **Tutte-Grothendieck ring of matroids** to be the commutative ring $K_0(\mathrm{Mat})$ defined as the free abelian group on isomorphism classes of matroids, together with multiplication given by the direct sum of matroids, modulo the relations that if e is neither a loop nor a coloop of M then $[M] = [M \backslash e] + [M/e]$.

If R is a commutative ring, an R-valued **Tutte-Grothendieck invariant** is a homomorphism from $K_0(\mathrm{Mat})$ to R. The following result due to Crapo and Brylawski asserts that the Tutte polynomial is the universal Tutte-Grothendieck invariant:

Theorem 3.7.
(1) *The Tutte polynomial is the unique Tutte-Grothendieck invariant* $T : K_0(\mathrm{Mat}) \to \mathbb{Z}[x,y]$ *satisfying* $T(\mathrm{coloop}) = x$ *and* $T(\mathrm{loop}) = y$.
(2) *More generally, if* $\phi : K_0(\mathrm{Mat}) \to R$ *is any Tutte-Grothendieck invariant then* $\phi = \phi_0 \circ T$ *where* $\phi_0 : \mathbb{Z}[x,y] \to R$ *is the unique ring homomorphism sending* x *to* $\phi(\mathrm{coloop})$ *and* y *to* $\phi(\mathrm{loop})$.

Similarly, the characteristic polynomial is the universal Tutte-Grothendieck invariant for combinatorial geometries. More precisely, if ϕ is any Tutte-Grothendieck invariant such that $\phi(M) = \phi(\hat{M})$ for every loopless matroid M, then

$$\phi(M) = (-1)^{r(M)} \chi_M(1 - \phi(\mathrm{coloop})).$$

The Tutte polynomial has a number of remarkable properties. For example, one has the following compatibility with matroid duality:

$$T_M(x,y) = T_{M^*}(y,x).$$

3.5. **The rank polynomial.** Let M be a simple matroid with lattice of flats L. The **rank polynomial** of M is

$$\rho_M(t) = \sum_{F \in L} t^{r(M)-r(F)} = W_0(M)t^r + W_1(M)t^{r-1} + \cdots + W_r(M).$$

The coefficients $W_k(M)$ of $\rho_M(t)$ are strictly positive, and are called the **Whitney numbers of the second kind**. Concretely, $W_k(M)$ is the number of flats in M of rank k. Comparing with (3.1), we see that the coefficients of $\chi_M(t)$ and $\rho_M(t)$ are related by

$$w_k(M) = \sum_{F \in L \,:\, r(F)=k} \mu_L(\emptyset, F),$$

$$W_k(M) = \sum_{F \in L \,:\, r(F)=k} 1.$$

For the matroid $M_n := M(K_n)$ associated to the complete graph K_n, $w_k(M_n) = s(n,k)$ and $W_k(M_n) = S(n,k)$ are the Stirling numbers of the first and second kind, respectively (hence the name for the Whitney numbers).

It is conjectured that the Whitney numbers of the second kind form a log-concave, and hence unimodal, sequence for every simple matroid M. This, however, remains an open problem.

It is a recent theorem of Huh–Wang [14] that if M is a rank r matroid which is representable over some field, then $W_1 \le W_2 \le \cdots \le W_{\lfloor r/2 \rfloor}$ and $W_k \le W_{r-k}$ for every $k \le r/2$, see §4.10 below.

3.6. **Motivic interpretation of the characteristic polynomial.** Let k be a field. The **Grothendieck ring of k-varieties** is the commutative ring $K_0(\mathrm{Var}_k)$ defined as the free abelian group on isomorphism classes of k-varieties, together with multiplication given by the product of varieties, modulo the "scissors congruence" relations that whenever $Z \subset X$ is a closed k-subvariety we have $[X] = [X \backslash Z] + [Z]$.

When $k = \mathbb{C}$ or $k = \mathbb{F}_q$ is a finite field, there is a canonical ring homomorphism[3] $e : K_0(\mathrm{Var}_k) \to \mathbb{Z}[t]$ with the property that $e(\mathbb{A}^1_k) = t$.

Let $V \subset k^m$ be an r-dimensional subspace representing a matroid M with lattice of flats L. With the notation of Example 2.8, an elementary inclusion-exclusion argument shows that in the ring $K_0(\mathrm{Var}_k)$ we have the "motivic" identity

$$[V \cap (k^\times)^m] = \sum_{F \in L} \mu_L(0_L, F)[V \cap L_F]. \tag{3.3}$$

(For example, if V is a *generic* subspace of k^m then

$$[V \cap (k^\times)^m] = [V \cap L_\emptyset] - \sum_i [V \cap L_i] + \sum_{|I|=2} [V \cap L_I] - \cdots,$$

but in general there are subspace relations between the various $V \cap L_I$ governed by the combinatorics of the underlying matroid.)

The identity (3.3), which is strongly reminiscent of (3.1), can be used to establish Theorem 3.5 in the representable case. Since $e(V \cap L_F) = t^{r-r(F)}$, it also explains the theorem of Orlik and Solomon that for $k = \mathbb{C}$ the Hodge polynomial of $V \cap (\mathbb{C}^*)^m$ is $\chi_M(t)$, as well as the theorem of Athanasiadis that for $k = \mathbb{F}_q$ we have

$$|V \cap (\mathbb{F}_q^\times)^m| = \chi_M(q).$$

We mentioned in §3.3 that the reduced characteristic polynomial $\bar{\chi}_M(t)$ is the "projective" analogue of $\chi_M(t)$. A concrete way to interpret this statement in the

[3]This homomorphism may be defined when $k = \mathbb{C}$ as the compactly supported χ_y-genus from mixed Hodge theory, and when $k = \mathbb{F}_q$ as $[X] \mapsto |X(\mathbb{F}_q)|$ (which coincides with the compactly supported χ_y-genus in ℓ-adic cohomology).

representable case is that since $k^\times$, which satisfies $e(k^\times) = t-1$, acts freely on $V \cap (k^\times)^m$, we have

$$e\left(\mathbb{P}(V \cap (k^\times)^m)\right) = e(V \cap (k^\times)^m)/e(k^\times) = \chi_M(t)/(t-1) = \bar{\chi}_M(t).$$

4. Overview of the proof of the Rota–Welsh Conjecture

We briefly outline the strategy used by Adiprasito, Huh, and Katz in their proof of the Rota–Welsh conjecture. The first step is to define a *Chow ring* $A^*(M)$ associated to an arbitrary loopless matroid M. The definition of this ring is motivated by work of Feichtner and Yuzvinsky [9], who noted that when M is realizable over $\mathbf{C}$, the ring $A^*(M)$ coincides with the usual Chow ring of the de Concini–Procesi "wonderful compactification" Y_M of the hyperplane arrangement complement associated to M [6, 7]. (Although the definition of $A^*(M)$ is purely combinatorial and does not require any notions from algebraic geometry, it would presumably be rather hard to motivate the following definition without knowing something about the relevant geometric background.) Note that Y_M is a smooth projective variety of dimension $d := r-1$, where r is the rank of M.

4.1. **The Chow ring of a matroid.** Let M be a loopless matroid, and let $\mathcal{F}' = \mathcal{F} \backslash \{\emptyset, E\}$ be the poset of *non-empty proper flats* of M. The graded ring $A^*(M)$ is defined as the quotient of the polynomial ring $S_M = \mathbf{Z}[x_F]_{F \in \mathcal{F}'}$ by the following two kinds of relations:

- (CH1) For every $a, b \in E$, the sum of the x_F for all F containing a equals the sum of the x_F for all F containing b.
- (CH2) $x_F x_{F'} = 0$ whenever F and F' are incomparable in the poset $\mathcal{F}'$.

The generators x_F are viewed as having degree one. There is an isomorphism $\deg : A^d(M) \to \mathbf{Z}$ determined uniquely by the property that $\deg(x_{F_1} x_{F_2} \cdots x_{F_d}) = 1$ whenever $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_d$ is a maximal flag in $\mathcal{F}'$.

It may be helpful to note that $A^*(M)$ can be naturally identified with equivalence classes of piecewise polynomial functions on the Bergman fan Σ_M. The fact that there is a unique homomorphism $\deg : A^d(M) \to \mathbf{Z}$ as above means, in the language of tropical geometry, that there is a unique (up to scalar multiple) set of integer weights on the top-dimensional cones of Σ_M which make it a *balanced* polyhedral complex.

4.2. **Connection to Hodge Theory.** If M is realizable, one can use the so-called *Hodge-Riemann relations* from algebraic geometry, applied to the smooth projective algebraic variety Y_M whose Chow ring is $A^*(M)$, to prove the Rota–Welsh log-concavity conjecture for M. This is (in retrospect, anyway) the basic idea in the earlier paper of Huh and Katz, about which we will say more in §4.8 below.

We now quote from the introduction to [1]:

> "While the Chow ring of M could be defined for arbitrary M, it was unclear how to formulate and prove the Hodge-Riemann relations... We are nearing a difficult chasm, as there is no reason to expect a working Hodge theory beyond the case of realizable matroids. Nevertheless, there was some evidence on the existence of such a theory for arbitrary matroids."

What the authors of [1] do is to formulate a purely combinatorial analogue of the Hard Lefschetz Theorem and Hodge-Riemann relations and prove them for the ring $A^*(M)_{\mathbf{R}} := A^*(M) \otimes \mathbf{R}$ in a purely combinatorial way, making no use of algebraic geometry. The idea is that although the ring $A^*(M)_{\mathbf{R}}$ is not actually the cohomology ring of a smooth projective variety, from a Hodge-theoretic point of view it behaves as if it were.

4.3. **Ample classes, Hard Lefschetz, and Hodge–Riemann.** In order to formulate precisely the main theorem of [1], we need a combinatorial analogue of hyperplane classes, or more generally of ample and nef divisors. The connection goes through strictly submodular functions. A function $c : 2^E \to \mathbf{R}_{\geq 0}$ is called **strictly submodular** (resp. **submodular**) if $c(E) = c(\emptyset) = 0$ and $c(A \cup B) + c(A \cap B) < c(A) + c(B)$ (resp. $c(A \cup B) + c(A \cap B) \leq c(A) + c(B)$) whenever A, B are incomparable subsets of E. Strictly submodular functions exist, and each submodular c gives rise to an element $\ell(c) = \sum_{F \in \mathcal{F}'} c(F) x_F \in A^1(M)_{\mathbb{R}}$. The convex cone of all $\ell(c) \in A^1(M)_{\mathbb{R}}$ associated to strictly submodular (resp. submodular) classes is called the **ample cone**[4] (resp. **nef cone**).

Ample (resp. nef) classes in $A^1(M)_{\mathbb{R}}$ correspond in a natural way to strictly convex (resp. convex) piecewise-linear functions on the Bergman fan Σ_M (cf. §2.9).

The main theorem of [1] is the following:

Theorem 4.1 (Adiprasito–Huh–Katz, 2015)**.** *Let M be a matroid of rank $r = d+1$, let $\ell \in A^1(M)_{\mathbf{R}}$ be ample, and let $0 \leq k \leq \frac{d}{2}$. Then:*

(1) *(Poincaré duality) The natural multiplication map gives a perfect pairing $A^k(M) \times A^{d-k}(M) \to A^d(M) \cong \mathbf{Z}$.*
(2) *(Hard Lefschetz Theorem) Multiplication by ℓ^{d-2k} determines an isomorphism $L_\ell^k : A^k(M)_{\mathbf{R}} \to A^{d-k}(M)_{\mathbf{R}}$.*
(3) *(Hodge-Riemann relations) The natural bilinear form*
$$Q_\ell^k : A^k(M)_{\mathbf{R}} \times A^k(M)_{\mathbf{R}} \to \mathbf{R}$$
defined by $Q_\ell^k(a, b) = (-1)^k a \cdot L_\ell^k b$ is positive definite on the kernel of $\ell \cdot L_\ell^k$ (the so-called "primitive classes").

This is all in very close analogy with analogous results in classical Hodge theory.

4.4. **Combinatorial Hodge theory implies the Rota–Welsh Conjecture.** To see why the Theorem 4.1 implies the Rota–Welsh conjecture, fix $e \in E = \{0, \ldots, n\}$. Let $\alpha(e) \in S_M$ be the sum of x_F over all F containing e, and let $\beta(e) \in S_M$ be the sum of x_F over all F not containing e. The images of $\alpha(e)$ and $\beta(e)$ in $A^1(M)$ do not depend on e, and are denoted by α and β, respectively.

Theorem 4.2. *Let $\bar{\chi}_M(t) := \chi_M(t)/(t-1)$ be the reduced characteristic polynomial of M, and write $\bar{\chi}_M(t) = m_0 t^d - m_1 t^{d-1} + \cdots + (-1)^d m_d$. Then $m_k = \deg(\alpha^{d-k} \cdot \beta^k)$ for all $k = 0, \ldots, d$.*

The proof of this result is based on Weisner's theorem and the following (positive) combinatorial formula for the coefficient m_k. A k-step flag $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ in $\mathcal{F}'$ is said to be **initial** if $r_M(F_i) = i$ for all i, and **descending** if
$$\min(F_1) > \min(F_2) > \cdots > \min(F_k) > 0,$$
where for $F \subseteq \{0, 1, \ldots, n\}$ we set $\min(F) = \min\{i \ : \ i \in F\}$.

Proposition 4.3. *m_k is the number of initial, descending k-step flags in $\mathcal{F}'$.*

[4] Actually, the ample cone in [1] is *a priori* larger than what we've just defined, but this subtlety can be ignored for the present purposes.

Although α and β are not ample, they belong to the nef cone and one may view them as a limit of ample classes. This observation, together with the Hodge-Riemann relations for $A^0(M)$ and $A^1(M)$ and Theorem 4.2, allows one to deduce the Rota–Welsh conjecture in a formal way.

4.5. **Log-concavity of f-vectors of matroids.** The Rota–Welsh conjecture implies a conjecture of Mason and Welsh on f-vectors of matroids.

Corollary 4.4 (Mason–Welsh Conjecture)**.** *Let M be a matroid on E, and let $f_k(M)$ be the number of independent subsets of E with cardinality k. Then the sequence $f_k(M)$ is log-concave and hence unimodal.*

To deduce Corollary 4.4 from the results of [1], one proceeds by showing that the signed f-polynomial

$$f_0(M)t^r - f_1(M)t^{r-1} + \cdots + (-1)^r f_r(M)$$

of the rank r matroid M coincides with the reduced characteristic polynomial of an auxiliary rank $r+1$ matroid M' constructed from M, the so-called *free co-extension of M*.[5] This identity was originally proved by Brylawski [3] and subsequently rediscovered by Lenz [16].

4.6. **High-level overview of the strategy for proving Theorem 4.1.** The main work in [1] is of course establishing Poincaré duality and especially the Hard Lefschetz Theorem and Hodge–Riemann relations for M. The proof is motivated by Peter McMullen's observation [17] that one can reduce the so-called "g-conjecture"[6] for arbitrary simple polytopes to the case of simplices using the "flip connectivity" of simple polytopes of given dimension. (In the case of a simplex, the Hard Lefschetz theorem and the Hodge–Riemann relations can be directly verified "by hand".)

A key observation in [1] motivated by McMullen's work is that for any two matroids M and M' of the same rank on the same ground set E, there is a diagram

$$\Sigma_{\mathrm{M}} \xrightarrow{\text{"}flip\text{"}} \Sigma_1 \xrightarrow{\text{"}flip\text{"}} \Sigma_2 \xrightarrow{\text{"}flip\text{"}} \cdots \xrightarrow{\text{"}flip\text{"}} \Sigma_{\mathrm{M}'}\ ,$$

where each matroidal flip preserves the validity of the Hard Lefschetz Theorem and Hodge-Riemann relations.[7] Using this, one reduces Theorem 4.1 to the case of uniform matroids (cf. Example 2.2), where the assertions in Theorem 4.1 can be directly verified "by hand".

The inductive approach to the hard Lefschetz theorem and the Hodge-Riemann relations in [1] is modeled on the observation that any facet of a permutohedron is the product of two smaller permutohedrons.

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[5]To define the free co-extension, let e be an auxiliary element not in E and let $E' = E \cup \{e\}$. The **free extension of** M **by** e is the matroid $M + e$ on E' whose independent sets are the independent sets of M together with all sets of the form $I \cup \{e\}$ with I an independent set of M of cardinality at most $r-1$. The **free co-extension of** M **by** e is the matroid $M \times e$ on E' given by $M \times e = (M^* + e)^*$.

[6]For an overview of the g-conjecture and applications of Hodge theory to the enumerative geometry of polytopes, see e.g. Richard Stanley's article [20].

[7]A subtlety is that the intermediate objects Σ_i are balanced weighted rational polyhedral fans but not necessarily tropical linear spaces associated to some matroid. So one leaves the world of matroids in the course of the proof, unlike with McMullen's case of polytopes.

4.7. **Remarks on Chow equivalence.** The Chow ring $A^*(M)$ of a rank $d+1$ matroid M on $\{0,\dots,n\}$ coincides with the Chow ring of the smooth but *non-complete* toric variety $X(\Sigma_M)$ associated to the Bergman fan of M. One of the subtleties here, and one of the remarkable aspects of the results in [1], is that although the n-dimensional toric variety $X(\Sigma_M)$ is not complete, its Chow ring "behaves like" the Chow ring of a d-dimensional smooth projective variety.

When M is representable over a field k, there is a good reason for this: one can construct a map from a smooth projective variety Y of dimension d to $X(\Sigma_M)$ which induces (via pullback) an isomorphism of Chow rings

$$A^*(X(\Sigma_M)) \xrightarrow{\sim} A^*(Y).$$

For example, if $M = U_{2,3}$ is the uniform matroid represented over $\mathbb{C}$ by a line $\ell \subset \mathbb{P}^2$ in general position, its Bergman fan Σ_M is a tropical line in $\mathbb{R}^2$ (cf. Example 2.13) and the corresponding toric variety $X(\Sigma_M)$ is isomorphic to $\mathbb{P}^2\backslash\{0,1\infty\}$. Pullback along the inclusion map $\mathbb{P}^1 \cong \ell \hookrightarrow \mathbb{P}^2\backslash\{0,1,\infty\}$ induces a Chow equivalence between $\mathbb{P}^2\backslash\{0,1,\infty\}$ and $\mathbb{P}^1$. (However, that the induced map $H^*(\mathbb{P}^2\backslash\{0,1,\infty\},\mathbb{C}) \to H^*(\mathbb{P}^1,\mathbb{C})$ on singular cohomology rings is far from being an isomorphism.)

When M is not realizable, however, there is provably no such Chow equivalence between $A^*(M)$ and the Chow ring of a smooth projective variety Y mapping to $X(\Sigma_M)$ [1, Theorem 5.12].

The construction of Y in the realizable case follows from the theory of de Concini–Procesi "wonderful compactifications". One takes the toric variety $X(\Sigma_U)$ associated to the n-dimensional permutohedron P_n (cf. §2.9) – the so-called **permutohedral variety**[8] – and views the Bergman fan Σ_M of the realizable rank $d+1$ matroid M as a d-dimensional subfan of the normal fan Σ_U to P_n, which is a complete n-dimensional fan in $\mathbb{R}^n$. This induces an open immersion of toric varieties $X(\Sigma_M) \subset X(\Sigma_U)$, and the wonderful compactification Y of the hyperplane arrangement complement realizing M, which is naturally a closed subvariety of $X(\Sigma_U)$, belongs to the open subset $X(\Sigma_M)$. The induced inclusion map $Y \hookrightarrow X(\Sigma_M)$ realizes the desired Chow equivalence.

In this case, the linear relations (CH1) come from linear equivalence on the ambient permutohedral toric variety $X(\Sigma_U)$, pulled back along the open immersion $X(\Sigma_M) \hookrightarrow X(\Sigma_U)$, and the quadratic relations (CH2) come from the fact that if F and F' are incomparable flats then the corresponding divisors are disjoint in $X(\Sigma_U)$.

4.8. **Proof of log-concavity in the realizable case d'après Huh–Katz.** The geometric motivation for several parts of the proof of the Rota–Welsh Conjecture comes from the proof of the representable case given in [13], and is intimately connected with the geometry of the permutohedral variety. (We remind the reader, however, that asymptotically 100% of all matroids are not representable over any field [18].) We briefly sketch the argument from [13].

The n-dimensional permutohedral variety $X(\Sigma_U)$ is a smooth projective variety which can be considered as an iterated blow-up of $\mathbb{P}^n$. After fixing homogenous coordinates on $\mathbb{P}^n$, we get a number of distinguished linear subspaces of $\mathbb{P}^n$, for example the $n+1$ points having all but one coordinate equal to zero. We also get the coordinate lines between any two of those points, and in general we can

[8] The permutohedral variety is an example of a *Losev–Manin moduli space*.

consider all linear subspaces of the form $\bigcap_{i\in I} H_i$ where H_i is the i^{th} coordinate hyperplane and $I \subset E := \{0,1,\ldots,n\}$. The permtohedral variety $X(\Sigma_U)$ can be constructed by first blowing up the $n+1$ coordinate points, then blowing up the proper transforms of the coordinate lines, then blowing up the proper transforms of the coordinate planes, and so on. In particular, this procedure determines a distinguished morphism $\pi_1 : X(\Sigma_U) \to \mathbb{P}^n$ which is a proper modification of $\mathbb{P}^n$.

There is another distinguished morphism $\pi_2 : X(\Sigma_U) \to \mathbb{P}^n$ which can be obtained by composing π_1 with the standard Cremona transform $\mathrm{Crem} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given in homogeneous coordinates by $(x_0 : \cdots : x_n) \mapsto (x_0^{-1} : \cdots : x_n^{-1})$. Although Crem is only a rational map on $\mathbb{P}^n$, it extends to an automorphism of $X(\Sigma_U)$, i.e., there is a morphism $\widetilde{\mathrm{Crem}} : X(\Sigma_U) \to X(\Sigma_U)$ such that $\pi_1 \circ \widetilde{\mathrm{Crem}} = \mathrm{Crem} \circ \pi_1$ as rational maps $X(\Sigma_U) \dashrightarrow \mathbb{P}^n$. In other words, $\widetilde{\mathrm{Crem}} : X(\Sigma_U) \to X(\Sigma_U)$ resolves the indeterminacy locus of Crem. We set $\pi_2 = \pi_1 \circ \widetilde{\mathrm{Crem}}$.

A rank $d+1$ loopless matroid M on E which is representable over k corresponds to a $(d+1)$-dimensional subspace V of k^{n+1} which is not contained in any hyperplane. Let $\mathbb{P}(V) \subset \mathbb{P}^n$ be the projectivization of V. Like $X(\Sigma_U)$ itself, the proper transform $\widetilde{\mathbb{P}(V)}$ of $\mathbb{P}(V)$ in $X(\Sigma_U)$ can be constructed as an iterated blowup, in this case a blowup of $\mathbb{P}(V)$ at its intersections with the various coordinate spaces of $\mathbb{P}^n$. In fact, $\widetilde{\mathbb{P}(V)}$ coincides with the de Concini–Procesi wonderful compactification Y mentioned above. The homology class of $\widetilde{\mathbb{P}(V)}$ in the permutohedral variety depends only on the matroid M, and not on the particular choice of the subspace V. We denote by p_1, p_2 the restrictions to $\widetilde{\mathbb{P}(V)}$ of π_1, π_2, respectively.

The key fact from [13] linking $\widetilde{\mathbb{P}(V)}$ and the ambient permutohedral variety to the Rota–Welsh Conjecture is the following (compare with Theorem 4.2):

Theorem 4.5. *Let H be the class of a hyperplane in* $\mathrm{Pic}(\mathbb{P}^n)$*, let $\alpha = p_1^{-1}(H)$, and let $\beta = p_2^{-1}(H)$. Then:*

(1) *The class of $(p_1 \times p_2)(\widetilde{\mathbb{P}(V)})$ in the Chow ring of $\mathbb{P}^n \times \mathbb{P}^n$ is*

$$m_0[\mathbb{P}^d \times \mathbb{P}^0] + m_1[\mathbb{P}^{d-1} \times \mathbb{P}^1] + \cdots + m_r[\mathbb{P}^0 \times \mathbb{P}^d].$$

(2) *The k^{th} coefficient m_k of the reduced characteristic polynomial $\bar{\chi}_M(t)$ is equal to* $\deg(\alpha^{d-k}\beta^k)$.

The Rota–Welsh conjecture for representable matroids follows immediately from Theorem 4.5(2) and the Khovanskii–Teissier inequality, which says that if X is a smooth projective variety of dimension d and α, β are nef divisors on X then $\deg(\alpha^{d-k}\beta^k)$ is a log-concave sequence.

4.9. **The Kähler Package.** The proof of the Khovanskii–Teissier inequality uses Kleiman's criterion to reduce to the case where α, β are ample, then uses the Kleiman-Bertini theorem to reduce to the case of surfaces, in which case the desired inequality is precisely the classical Hodge Index Theorem. The Hodge Index Theorem itself is a very special case of the Hodge–Riemann relations.

One of the original approaches by Huh and Katz to extend their work to non-representable matroids was to try proving a tropical version of the Hodge Index

Theorem for surfaces. However, Huh found a counterexample to the naive formulation of this result[9], and the situation appears quite delicate — it is unclear what the hypotheses for a tropical Hodge Index Theorem should be and how to reduce the desired inequalities to this special case. So instead, inspired by the work of Peter McMullen, Huh and Katz (together with Adiprasito) took a rather different approach.

In both the realizable case from [13] and the general case from [1], one needs only a very special case of the Hodge–Riemann relations to deduce log-concavity of the coefficients of $\bar{\chi}_M(t)$.[10] And Poincaré Duality and the Hard Lefschetz Theorem for Chow rings of matroids are not needed at all for this application. So it's reasonable to wonder whether Theorem 4.1 is overkill if one just wants a proof of the Rota–Welsh conjecture. However, the situation is more delicate than it might appear: in practice, Poincaré Duality, the Hard Lefschetz Theorem, and the Hodge–Riemann relations tend to come bundled together in what June Huh has dubbed the *Kähler package*. This is the case, for example, in the de Cataldo–Migliorini approach to Hodge theory [4], in the work of McMullen and Fleming–Karu on Hodge theory for simple polytopes [17, 11], in the work of Elias–Williamson [8], and in the paper of Adiprasito–Huh–Katz [1].

In the case of simple polytopes and the g-conjecture, what is needed is in fact the Hard Lefschetz Theorem, and not the Hodge–Riemann relations, for the appropriate Chow ring. But again the proof proceeds by establishing the full Kähler package. The following excerpt from the Introduction to [11] may be helpful in summarizing why:

> "The next step in McMullen's proof is to show that in fact the Hodge–Riemann–Minkowski relations for all pairs of complete simplicial fans and convex functions in dimension $n-1$ imply the Hard Lefschetz theorem for the same in dimension n. Thus, by induction we may assume that a convex function ℓ defines a Lefschetz operation on $H(\Sigma)$ or, equivalently, that the form Q_ℓ is nondegenerate. This implies that for a continuous family of convex functions ℓ_t on Σ, the signature of Q_{ℓ_t} is the same for all t, and therefore if the Hodge–Riemann–Minkowski relations hold for one function, they hold for every function in the family. The last step of McMullen's proof is to study the change in the signature as a fan undergoes an "elementary flip". Any complete simplicial Σ with convex function ℓ can be transformed to the normal fan of a simplex (for which direct calculation is possible) by a sequence of such flips, with continuous deformations of associated convex functions. So the proof is finished by showing that the Hodge–Riemann–Minkowski relations hold on one side of the flip if and only if they hold on the other side. This is done by explicitly relating the signatures of the forms Q_{ℓ_t}."

In rough outline, the inductive procedure in [1] is quite similar to the one described above. In fact, according to June Huh (private communication): "Following the argument of [1], the Hard Lefschetz Theorem and Hodge-Riemann relations for simple polytopes can be obtained as an (important) special case." One of the

[9]Huh's counterexample to the tropical Hodge Index Theorem is closely related to the counterexample given in [2] to the strongly positive Hodge Conjecture of Demailly.

[10]Presumably one can use the general Hodge–Riemann relations to deduce other combinatorial facts of interest about matroids!

important differences between [11] and [1], already mentioned above, is that the intermediate objects in the inductive procedure from [1], obtained by applying flips to Bergman fans of matroids, are no longer themselves Bergman fans of matroids (whereas in [11] all of the simplicial fans which appear come from simple polytopes). The main difference, however, between the proofs in [1] and their polytopal counterparts in [11] is that in the polytope case one is working with n-dimensional fans in $\mathbb{R}^n$, whereas in the matroid case one is working with d-dimensional fans in $\mathbb{R}^n$, where $d < n$ except in the trivial (but important) case of the n-dimensional permutohedral fan. In both the polytope and matroid situations the fan in question defines an n-dimensional toric variety, but the toric variety is projective in the polytope case and non-complete in the matroid case. As mentioned above in §4.7, the "miracle" in the matroid case is that the Chow ring of the n-dimensional non-complete toric variety $X(\Sigma_M)$ behaves as if it were the Chow ring of a d-dimensional smooth projective variety; in particular, it satisfies Poincaré Duality, Hard Lefschetz, and Hodge–Riemann of "formal" dimension d.

4.10. **Whitney numbers of the second kind.** The Whitney numbers of the second kind $W_k(M)$ (cf. §3.5) are much less tractable then their first-kind counterparts. In particular, the log-concavity conjecture for them remains wide open. However, there has been recent progress by Huh and Wang [14] concerning a related conjecture, the so-called "top-heavy conjecture" of Dowling and Wilson:

Conjecture 4.6. *Let M be a matroid of rank r. Then for all $k < r/2$ we have $W_k(M) \leq W_{r-k}(M)$.*

In analogy with the work of Huh–Katz, Huh and Wang prove:

Theorem 4.7 (Huh–Wang, 2016)**.** *For all matroids M representable over some field k:*

(1) *The first half of the sequence of Whitney numbers of the second kind is unimodal, i.e., $W_1(M) \leq W_2(M) \leq \cdots \leq W_{\lfloor r/2 \rfloor}(M)$.*
(2) *Conjecture 4.6 is true.*

The following corollary is a generalization of the de Bruijn-Erdös theorem that every non-collinear set of points E in a projective plane determines at least $|E|$ lines:

Corollary 4.8. *Let V be a d-dimensional vector space over a field and let E be a subset which spans V. Then (in the partially ordered set of subspaces spanned by subsets of E), there are at least as many $(d-k)$-dimensional subspaces as there are k-dimensional subspaces, for every $k \leq d/2$.*

We will content ourselves with just a couple of general remarks concerning the proof of Theorem 4.7. Unlike in the Rota–Welsh situation of Whitney numbers of the first kind, the projective algebraic variety Y'_M which one associates to M in this case is highly singular; thus instead of invoking the Kähler package for smooth projective varieties, Huh and Wang have to use analogous but much harder results about intersection cohomology. Specifically, they require the Bernstein–Beilinson–Deligne–Gabber decomposition theorem for intersection complexes[11] and the Hard Lefschetz theorem for ℓ-adic intersection cohomology of projective varieties.

[11]See [5] for an overview of the decomposition theorem and its many applications.

It is tempting to fantasize about a proof of Conjecture 4.6 along the lines of [1]. According to June Huh (private correspondence), the first significant challenge in this direction is to construct a combinatorial model for intersection cohomology of the variety Y'_M.

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TAO'S RESOLUTION OF THE ERDŐS DISCREPANCY PROBLEM

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The Erdős discrepancy problem is an easily stated question about arbitrary functions f from the positive integers to ± 1. It asks whether the signs ± 1 can be arranged evenly over all subsequences of the form kj for a given $k \in \mathbb{N}$ and as j varies. Precisely, must it always be the case that

$$\sup_{k,n\in\mathbb{N}} \Big| \sum_{j=1}^{n} f(kj) \Big| = \infty? \tag{1}$$

The question appears in many of Erdős's lists of unsolved problems [5, 4, 3], and in [4] he dates the conjecture to the 1930's. Erdős highlights a striking special case of the problem: Suppose f is a completely multiplicative function (that is, $f(mn) = f(m)f(n)$ for all natural numbers m and n) taking the values ± 1, then is it true that

$$\sup_{n\in\mathbb{N}} \Big| \sum_{j=1}^{n} f(j) \Big| = \infty? \tag{2}$$

Since $f(kj) = f(k)f(j)$ in this situation, clearly a positive solution to problem (1) implies a resolution of problem (2) as well. So as not to keep the reader in suspense, let us state at once that the Erdős discrepancy problem was answered affirmatively by Tao [19]:

Theorem 1 (Tao)**.** *For any function $f : \mathbb{N} \to \{-1, 1\}$ we have*

$$\sup_{k,n\in\mathbb{N}} \Big| \sum_{j=1}^{n} f(kj) \Big| = \infty.$$

In particular, the partial sums of a completely multiplicative function taking the values ± 1 are unbounded; that is, (2) holds.

The Erdős discrepancy problem asks whether every two coloring of the natural numbers must exhibit some disorder when viewed along homogenous arithmetic progressions jk (for $1 \le j \le n$). One may weaken the problem to allow all arithmetic progressions $a+jk$: is it true that for all $f : \mathbb{N} \to \{-1, 1\}$,

$$\sup_{a,k,n\in\mathbb{N}} \Big| \sum_{j=1}^{n} f(a + jk) \Big| = \infty? \tag{3}$$

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Roth [17] established the existence of such irregularities of distribution. Indeed, more generally he showed that if A is a subset of the integers up to N with $|A| = \rho N$ then there exists an absolute positive constant c such that

$$\sup_{\substack{k\le\sqrt{N}\\ a \mod k}} \Big| \sum_{\substack{n\in A\\ n\equiv a \mod k}} 1 - \frac{\rho N}{q}\Big| \ge c\sqrt{\rho(1-\rho)}N^{1/4}.$$

In other words, the only subsets of $[1, N]$ that are evenly distributed in all arithmetic progressions with moduli below $\sqrt{N}$ are (essentially) the empty set (with $\rho = 0$) or the whole set (with $\rho = 1$). The weak Erdős discrepancy problem (3) is only relevant for $\rho \approx 1/2$, and so Roth's theorem establishes (3), showing even a strong quantitative version. Matousek and Spencer [14] have shown that the $N^{1/4}$ bound in Roth's result is best possible.

The Erdős discrepancy problem for homogeneous progressions (with one fewer degree of freedom) has proved much more difficult, in part because the discrepancy can be much smaller than one might expect. A random sequence of ± 1 would exhibit disorder in its partial sums up to N typically on the scale of $\sqrt{N}$ (the central limit theorem), and occasionally on the scale of $\sqrt{N \log\log N}$ (the law of the iterated logarithm). In Roth's theorem, the discrepancy along all progressions is still of size a power of N, even if not $\sqrt{N}$. However, it is not hard to construct ± 1 sequences where the discrepancy along homogeneous progressions grows only logarithmically, and this slow rate of growth indicates why the Erdős discrepancy problem is so delicate.

Example 1. Consider the function

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \mod 3 \\ -1 & \text{if } n \equiv 2 \mod 3 \\ 0 & \text{if } n \equiv 0 \mod 3. \end{cases}$$

Then χ is a completely multiplicative function and periodic with period 3 – that is, it is a Dirichlet character mod 3 – and its partial sums are bounded (since every three consecutive terms sum to zero). Therefore the left sides of (1) and (2) are both finite. More generally one can take any quadratic character mod q (generalizations of the Legendre symbol mod p), to get examples of completely multiplicative functions that are also periodic and which have bounded partial sums. Of course, this is not a counterexample to (1) and (2) since χ takes the value 0 in addition to ± 1.

Example 2. A completely multiplicative function may be specified by its values on the primes. We tweak Example 1, defining the completely multiplicative function $\widetilde{\chi}$ by setting $\widetilde{\chi}(p) = \chi(p)$ if $p \neq 3$, and $\widetilde{\chi}(3) = 1$. Then for any k and n we have

$$\Big|\sum_{j=1}^{n} \widetilde{\chi}(kj)\Big| = \Big|\sum_{j=1}^{n} \widetilde{\chi}(j)\Big| = \Big|\sum_{\substack{\ell\ge 0\\ 3^\ell\le n}} \sum_{m\le n/3^\ell} \chi(m)\Big| \le 1 + \frac{\log n}{\log 3}.$$

On the other hand, taking $k = 1$ and $n = 1 + 3 + 3^2 + \ldots + 3^r$ we find

$$\Big|\sum_{j=1}^{n} \widetilde{\chi}(n)\Big| = \Big|\sum_{\ell=0}^{r} \sum_{m \le 3^{r-\ell}+1} \chi(m)\Big| = r + 1 = \Big\lceil \frac{\log n}{\log 3} \Big\rceil.$$

Thus, in this example, the discrepancy along homogeneous progressions does go to infinity, but only at a slow logarithmic pace. In [1], Borwein, Choi and Coons carry out an analysis of examples of this type, settling the Erdős discrepancy problem for modified characters $\widetilde{\chi}_p$ defined to be the Legendre symbol $\mod p$ on primes $\ell \neq p$, and with $\widetilde{\chi}_p(p) = 1$.

Example 3. Numerical examples. The sequence $(1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1)$ of length 11 has discrepancy 1 along homogeneous progressions, and is the longest such sequence [10]. In [9] it is shown that the longest sequence of ± 1 with discrepancy 2 along homogeneous progressions has size 1160, and that there is a sequence of length 13000 with discrepancy 3.

While the examples above indicate the delicate nature of the Erdős discrepancy problem, Tao's proof in fact gives the following more general result.

Theorem 2 (Tao)**.** *Let $\mathcal{H}$ be a Hilbert space, and let $f : \mathbb{N} \to \mathcal{H}$ be a function with $\|f(n)\|_{\mathcal{H}} = 1$ for all n. Then*

$$\sup_{k,n \in \mathbb{N}} \Big\| \sum_{j=1}^{n} f(jk) \Big\|_{\mathcal{H}} = \infty.$$

Example 4. The Hilbert space in Theorem 2 can be real or complex. In particular, Theorem 2 applies to any function f from the natural numbers to the unit circle $\{|z| = 1\}$. To specialize once more, rather than looking at ± 1 completely multiplicative functions (as in (2)) we may consider any completely multiplicative function $f : \mathbb{N} \to \{|z| = 1\}$ and ask whether

$$\sup_{n \in \mathbb{N}} \Big| \sum_{j=1}^{n} f(j) \Big| = \infty? \tag{4}$$

Of course Theorem 2 provides a positive answer to this question.

Example 5. In Theorem 2 the discrepancy can grow as slowly as $\sqrt{\log n}$. Take $\mathcal{H}$ to be a Hilbert space with orthonormal basis e_0, e_1, e_2, …. Write $n \in \mathbb{N}$ as $3^a b$ where a is a non-negative integer, and b is coprime to 3. Then set $f(n) = \chi(b) e_a$ where χ, the Dirichlet character $\mod 3$, is as in Example 1. Now if $k = 3^c d$ (with c a non-negative integer and d coprime to 3) then

$$\Big\| \sum_{j=1}^{n} f(kj) \Big\|_{\mathcal{H}} = \Big\| \sum_{\substack{a \ge 0 \\ 3^a \le n}} e_{c+a} \sum_{b \le n/3^a} \chi(b) \Big\|_{\mathcal{H}} \le \sqrt{1 + \log n / \log 3}.$$

As in Example 2, this bound is attained if $n = 1 + 3 + \ldots + 3^r$.

The Erdős discrepancy problem remained dormant for a long time, with no promising avenues of attack. In December 2009, Gowers proposed the Erdős discrepancy problem as a possible Polymath project, continuing his idea of "massively collaborative mathematics" which began earlier in 2009 with a new (and quantitative) proof of the density Hales-Jewett theorem (originally due to Furstenberg and Katznelson). Another successful polymath project is Polymath 8 from 2013, on improving the bounds on small gaps between primes after the breakthroughs of Zhang and Maynard-Tao. Polymath 5, the project on the Erdős discrepancy problem, began in January 2010, and was active until the end of 2012. Many interesting reformulations of the problem were found, and several of these are recounted in Gowers's article [6]. The website [16] documents the various Polymath discussions and blog posts centered around this problem.

For Tao's eventual resolution of the problem, two developments from the Polymath 5 project proved important: First, that the general Erdős discrepancy problem (in the form of general ± 1 sequences as in (1), or in the more general Hilbert space situation as in Theorem 2) can be deduced from the special case of completely multiplicative functions taking values on the unit circle (as in (4)). Second, if such a completely multiplicative function correlates with a Dirichlet character (in a sense to be made precise later) then the discrepancy does tend to infinity. In other words, if we are close to the situation of Example 2, then (generalizing the work of [1]) it is possible to show that the discrepancy must grow, even if only very slowly. Tao himself was a participant in the Polymath 5 project, and played a major role in both these developments.

The missing link is to show that completely multiplicative functions taking values on the unit circle and with bounded partial sums must correlate with characters. This is the most subtle part of Tao's argument, and is carried out in [20]. The starting point is a recent breakthrough of Matomäki and Radziwiłł [11] in understanding multiplicative functions in short intervals, with further important refinements due to Matomäki, Radziwiłł, and Tao [12]. On top of this, Tao brings to the problem several other novel ideas, some motivated by work in additive combinatorics.

In the rest of this article, I want to give an overview of the ideas behind Theorems 1 and 2, oversimplifying the situation to convey the flavor of the arguments.

Reducing the problem to completely multiplicative functions.

Suppose $\mathcal{H}$ is a Hilbert space, and $f(n)$ a sequence of unit vectors in $\mathcal{H}$ with bounded discrepancy:

$$\sup_{k,n} \Big\| \sum_{j=1}^{n} f(kn) \Big\| \le C. \tag{5}$$

From this we wish to construct completely multiplicative functions taking values on the unit circle and with small discrepancy; we won't exactly achieve this, but something good enough.

Let X be large, and let M be an integer much larger than X. Let $p_1, \ldots, p_r$ denote the primes below X. Define $F : (\mathbb{Z}/M\mathbb{Z})^r \to \mathcal{H}$, by setting

$$F(a_1, \ldots, a_r) = f(p_1^{a_1} \cdots p_r^{a_r}),$$

provided $0 \leq a_i \leq M - 1$, and then extending the definition periodically in each coordinate. Given a natural number n below X, write its prime factorization as $n = p_1^{a_1} \cdots p_r^{a_r}$ so that the exponents a_i are non-negative integers. Set $\pi(n) = (a_1, \ldots, a_r)$, which we view as an element of $(\mathbb{Z}/M\mathbb{Z})^r$.

If $j \leq X$ then note that the coordinates of $\pi(j)$ are all smaller than $\lceil \log X / \log 2 \rceil$. Therefore if $x = (x_1, \ldots, x_r)$ is such that $0 \leq x_i \leq M - X$ for all i then for all $n \leq X$ we have

$$\Big\| \sum_{j=1}^{n} F(x + \pi(j)) \Big\|_{\mathcal{H}} = \Big\| \sum_{j=1}^{n} f(j p_1^{x_1} \cdots p_r^{x_r}) \Big\|_{\mathcal{H}} \leq C. \tag{6}$$

Now there are M^r vectors $x \in (\mathbb{Z}/M\mathbb{Z})^r$, and the bound (6) applies to the vast majority of such x: namely for $(M - X + 1)^r = M^r + O(XM^{r-1})$ of them, and recall that M is much larger than X. For the few values of x not covered by (6), there holds the trivial bound

$$\Big\| \sum_{j=1}^{n} F(x + \pi(j)) \Big\|_{\mathcal{H}} \leq n \leq X.$$

Combining this trivial bound with the estimate (6), we conclude that for all $n \leq X$

$$\frac{1}{M^r} \sum_{x \in (\mathbb{Z}/M\mathbb{Z})^r} \Big\| \sum_{j=1}^{n} F(x + \pi(j)) \Big\|_{\mathcal{H}}^2 \leq C^2 + 1, \tag{7}$$

assuming that M is large enough compared with X.

With $e(t) = e^{2\pi i t}$, recall the Fourier transform on $(\mathbb{Z}/M\mathbb{Z})^r$: for $\xi \in (\mathbb{Z}/M\mathbb{Z})^r$ put

$$\widehat{F}(\xi) = \frac{1}{M^r} \sum_{x \in (\mathbb{Z}/M\mathbb{Z})^r} F(x) e\Big(-\frac{x \cdot \xi}{M} \Big),$$

which is an element of $\mathcal{H}$. The Fourier inversion formula gives

$$F(x) = \sum_{\xi \in (\mathbb{Z}/M\mathbb{Z})^r} \widehat{F}(\xi) e\Big(\frac{x \cdot \xi}{M} \Big),$$

and Parseval's formula reads

$$\sum_{\xi \in (\mathbb{Z}/M\mathbb{Z})^r} \|\widehat{F}(\xi)\|_{\mathcal{H}}^2 = \frac{1}{M^r} \sum_{x \in (\mathbb{Z}/M\mathbb{Z})^r} \|F(x)\|_{\mathcal{H}}^2 = 1. \tag{8}$$

Using the Fourier inversion formula

$$\sum_{j=1}^{n} F(x + \pi(j)) = \sum_{\xi \in (\mathbb{Z}/M\mathbb{Z})^r} \widehat{F}(\xi) e\Big(\frac{x \cdot \xi}{M} \Big) \Big(\sum_{j=1}^{n} e\Big(\frac{\pi(j) \cdot \xi}{M} \Big) \Big),$$

so that by Parseval and (7) we obtain

$$\sum_{\xi\in(\mathbb{Z}/M\mathbb{Z})^r} \|\widehat{F}(\xi)\|_{\mathcal{H}}^2 \Big| \sum_{j=1}^{n} e\Big(\frac{\pi(j)\cdot\xi}{M}\Big)\Big|^2 \le C^2+1. \tag{9}$$

We can now give a probabilistic interpretation of the estimate (9). We define a probability space of random completely multiplicative functions $\mathbf{f} : [1,X] \to \{|z|=1\}$, by setting $\mathbf{f}(p_j) = e(\xi_j/M)$ for all $1 \le j \le r$ with probability $\|\widehat{F}(\xi)\|_{\mathcal{H}}^2$ where $\xi = (\xi_1,\ldots,\xi_r) \in (\mathbb{Z}/M\mathbb{Z})^r$. Parseval's formula (8) shows that this is indeed a probability space. Further, the estimate (9) may be recast as

$$\mathbb{E}\Big(\Big|\sum_{j\le n} \mathbf{f}(j)\Big|^2\Big) \le C^2+1, \tag{10}$$

for all $n \le X$.

Thus for all large X, we have constructed a probability space (depending on X) of random completely multiplicative functions with values on the unit circle, such that the expected value of partial sums up to n (for all $n \le X$) is bounded. By a compactness argument, Tao shows that one can construct a space of random completely multiplicative functions such that (10) holds for all n instead of just n below X.

The estimate (10) forms the basis for Tao's proof of Theorem 2. Note that (10) is an average statement, and one cannot extract from it a (deterministic) completely multiplicative function with all partial sums up to X being bounded; for each n below X there clearly exists such a function, but perhaps no one function works for all $n \le X$. If we average (10) over all $n \le X$ then

$$\mathbb{E}\Big(\frac{1}{X}\sum_{n\le X}\Big|\sum_{j\le n} \mathbf{f}(j)\Big|^2\Big) \le C^2+1,$$

so that there must exist completely multiplicative functions f, taking values on the unit circle, such that

$$\frac{1}{X}\sum_{n\le X}\Big|\sum_{j\le n} f(j)\Big|^2 \le C^2+1.$$

We expect that this estimate cannot hold for large enough X, but this remains an open problem.

Let us highlight one interesting feature of this construction. Most values of $(a_1,\ldots,a_r)$ (with $0 \le a_i \le M-1$) will satisfy $a_i \ge A$ for all i and any fixed natural number A. Therefore most of the values of F will be built out of f evaluated at multiples of $\prod_{p\le X} p^A$. This shows the importance of needing all the values of f to be unit vectors; if f took small values along the multiples of some number (like in Example 1, or for any Dirichlet character) this will force F to be small almost all the time, and the construction leads nowhere.

Correlations of values of completely multiplicative functions.

Given a sequence of unit vectors in a Hilbert space $\mathcal{H}$ violating Theorem 2, the argument above produces a probability space of random completely multiplicative functions with partial sums bounded in expectation. For the sake of simplicity, let us suppose that we actually have a completely multiplicative function with values on the unit circle, and bounded partial sums; that is, a violation to (4).

To motivate the ideas that follow, we begin with a brief discussion of partial sums of multiplicative functions. Apart from the constant function 1, and Dirichlet characters (homomorphisms from $(\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$), perhaps the best known multiplicative functions are the Möbius function $\mu(n)$ and the Liouville function $\lambda(n)$. The Liouville function is completely multiplicative, and defined on the primes by $\lambda(p) = -1$; thus $\lambda(n) = 1$ if n has an even number of prime factors (counted with multiplicity) and -1 otherwise. The Möbius function $\mu(n)$ equals $\lambda(n)$ whenever n is square-free, and $\mu(n) = 0$ if n is divisible by the square of some prime; thus the Möbius function is merely multiplicative ($f(mn) = f(m)f(n)$ whenever m and n are coprime) rather than completely multiplicative. For the Möbius and Liouville functions the first main goal was to exhibit cancellation in their partial sums, since the estimates

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \mu(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \lambda(n) = 0 \tag{11}$$

can be elementarily shown to be equivalent to the prime number theorem. In fact we believe that $\lambda(n)$ behaves in many ways like a random collection of signs ± 1 (but there are limits to this belief), and that the partial sums above exhibit roughly square-root cancellation – this is related to the Riemann hypothesis. Of course our goal is to show that partial sums must get large sometimes; for the Möbius function (and a similar result holds for $\lambda(n)$), let us remark that the partial sums are known to be larger than $\sqrt{N}$ infinitely often, disproving a conjecture of Mertens (see [15]).

Given a completely multiplicative function f with $f(n) = \pm 1$, Erdős and Wintner asked whether

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} f(n)$$

always exists. If the limits in (11) exist, then it is not hard to show that they must be 0; so the Erdős–Wintner question offers another way of generalizing and thinking about the prime number theorem. Wintner [21] showed that if the function f is mostly like the constant function 1, then the limit above does exist; precisely, if

$$\sum_p \frac{1-f(p)}{p} < \infty, \qquad \text{then} \qquad \frac{1}{N}\sum_{n\le N} f(n) \to \prod_p \Big(1-\frac{f(p)}{p}\Big)^{-1}\Big(1-\frac{1}{p}\Big). \tag{12}$$

The harder direction (which includes the Liouville function) is to show that when the prime sum $\sum_p (1-f(p))/p$ diverges then the limiting average of $f(n)$ exists, and equals 0. This was beautifully resolved by Wirsing [22] who showed that

$$\sum_p \frac{1-f(p)}{p} = \infty \qquad \text{implies} \qquad \frac{1}{N}\sum_{n\le N} f(n) \to 0, \tag{13}$$

thereby settling this question of Erdős and Wintner.

For a complex valued completely multiplicative function f with $|f(n)| = 1$, the story is a little more complicated. The function $f(n) = n^{i\alpha}$ for a fixed real number α has no limiting average value: indeed, by comparing with the integral,

$$\frac{1}{N}\sum_{n\le N} n^{i\alpha} \sim \frac{1}{N}\frac{N^{1+i\alpha}}{1+i\alpha} = \frac{N^{i\alpha}}{1+i\alpha},$$

which oscillates with N. Halász [8] found the right generalization of Wirsing's work, and developed an ingenious analytic method to show that if

$$\sum_p \frac{1-\text{Re } f(p)p^{-i\alpha}}{p} = \infty \qquad \text{for all } \alpha\in\mathbb{R} \text{ then} \qquad \frac{1}{N}\sum_{n\le N} f(n)\to 0. \tag{14}$$

Given two multiplicative functions f and g taking values in the unit disc, it is convenient to define the distance between them (up to some point X) as

$$\mathbb{D}(f,g;X)^2 = \sum_{p\le X}\frac{1-\text{Re } f(p)\overline{g(p)}}{p},$$

and this satisfies the properties of a pseudo-metric, notably the triangle inequality; see [7] for further discussion. Thus the hypotheses in (13) and (14) may be interpreted as f not being close to the function 1 or to any function of the form $n^{i\alpha}$; in the language of [7], this is stated as f not pretending to be 1 or $n^{i\alpha}$, or (abusing the English language) as f being "unpretentious."

Let us now return to the Erdős discrepancy problem. Suppose f is a completely multiplicative function with $|f(n)| = 1$ and bounded partial sums. Since the partial sums are bounded, the sums $\sum_{h=1}^{H} f(n+h)$ must also be bounded for any interval $[n+1, n+H]$. Therefore one must have

$$\frac{1}{N}\sum_{n\le N}\Big|\sum_{h=1}^{H} f(n+h)\Big|^2 \le C,$$

for some constant C. Expanding the left side above gives

$$\begin{aligned}&\frac{1}{N}\sum_{n\le N}\sum_{h_1=h_2\le H}|f(n+h_1)|^2+\frac{1}{N}\sum_{n\le N}\sum_{h_1\ne h_2\le H}f(n+h_1)\overline{f(n+h_2)}\\ =&H+\sum_{h_1\ne h_2\le H}\frac{1}{N}\sum_{n\le N}f(n+h_1)\overline{f(n+h_2)}.\end{aligned}$$

If H is chosen to be large compared to C, then there must exist $h_1 \neq h_2 \leq H$ with

$$\frac{1}{N}\Big|\sum_{n\le N}f(n+h_1)\overline{f(n+h_2)}\Big|\ge\frac{1}{2H}. \tag{15}$$

Thinking of N as being very large compared to H, the above estimate says that there is a strong correlation between the values of f at n and $n+h$ with $h = h_2 - h_1 \neq 0$. This type of reasoning is often referred to as a van der Corput argument, originating in van der Corput's work on bounding exponential sums.

When can a multiplicative function f correlate with a shift of itself? Clearly the function $f(n) = 1$ does, and so does a function that pretends to be 1. More generally the function $n^{i\alpha}$, or anything pretending to be $n^{i\alpha}$, will correlate with its shifts: for large n there is not much difference between $n^{i\alpha}$ and $(n+1)^{i\alpha}$. Less obviously, if f is a Dirichlet character $\bmod q$, then the periodicity $\bmod q$ implies that $f(n)\overline{f(n+q)} = 1$ whenever $(n, q) = 1$. One can generalize this by taking $\chi(n)n^{i\alpha}$ for any real number α, and still more generally take any function pretending to be $\chi(n)n^{i\alpha}$. Note that $\widetilde{\chi}$ in Example 2 is a function pretending to be the character χ. In analogy with the results of Wirsing and Halász mentioned earlier, we may hope that these examples exhaust all the possibilities for a multiplicative function correlating with a shift of itself. Indeed Elliott [2] formulated such a conjecture – it needs a small technical correction, which is made in [12].

Unfortunately, very little is understood about correlations of multiplicative functions. For example, take the Liouville function $\lambda(n)$ which we said earlier is expected to look like a random sequence of signs ± 1. Furthermore, the Liouville function is known not to correlate with Dirichlet characters; this is essentially the prime number theorem in arithmetic progressions. Thus we should definitely expect that as $N \to \infty$

$$\frac{1}{N}\sum_{n\le N}\lambda(n)\lambda(n+1)\to 0, \tag{16}$$

which is also equivalent to saying that all four sign patterns $(+,+)$, $(+,-)$, $(-,+)$, $(-,-)$ occur roughly equally often among consecutive values of $\lambda(n)$. More generally, Chowla conjectured that if $a_j n + b_j$ (for $1 \leq j \leq k$) are affine functions with no two proportional to each other ($a_i b_j \neq b_i a_j$ for all $i \neq j$)

then as $N \to \infty$

$$\frac{1}{N} \sum_{n \le N} \lambda(a_1 n + b_1) \cdots \lambda(a_k n + b_k) \to 0.$$

Chowla's conjecture implies that among k consecutive values of λ, all 2^k possible patterns of signs occur equally frequently.

The first breakthrough toward these problems came with the work of Matomäki and Radziwiłł [11]. They showed that the Liouville function exhibits cancellation in almost all intervals $[n, n+h]$, as soon as $h \to \infty$ (however slowly). This was a remarkable advance on earlier results which required h to grow like a power of n. Toward (16) they established that for all large N

$$\Big| \sum_{n \le N} \lambda(n)\lambda(n+1) \Big| \le (1-\delta)N,$$

for some $\delta > 0$; from this it follows that all four sign patterns of $(\lambda(n), \lambda(n+1))$ appear a positive proportion of the time. Following this breakthrough, Matomäki, Radziwiłł and Tao [13] showed that all eight patterns of signs for $(\lambda(n), \lambda(n+1), \lambda(n+2))$ occur a positive proportion of the time, and moreover in [12] they showed an average version of the Chowla conjecture. While we have discussed just the Liouville function, the results in [11] and [12] apply more generally to all multiplicative functions. For an account of these papers, see my Bourbaki exposition [18].

These breakthroughs are still insufficient to show estimates like (16). However, Tao [20] realized that a weaker logarithmic version of (16) could be established. Namely, he showed that as $N \to \infty$

$$\frac{1}{\log N} \sum_{n \le N} \frac{1}{n} \lambda(n)\lambda(n+1) \to 0.$$

More generally, Tao showed a logarithmic version of the (corrected) Elliott conjecture: If f is a multiplicative function with $|f(n)| \le 1$ for all n and such that, for two affine functions $a_1 n + b_1$ and $a_2 n + b_2$ with $a_1 b_2 \ne a_2 b_1$,

$$\frac{1}{\log N} \Big| \sum_{n \le N} \frac{1}{n} f(a_1 n + b_1) \overline{f(a_2 n + b_2)} \Big|$$

is bounded away from 0, then there is a character χ to small modulus, and a small real number α such that $f(n)$ pretends to be $\chi(n) n^{i\alpha}$. This is the key result underlying Tao's proof of Theorems 1 and 2. It builds on the work of Matomäki, Radziwiłł and Tao – most notably their work in [12] – and adding several other novel ideas, particularly an "entropy decrement argument" reminiscent of ideas from additive combinatorics.

Coming back again to the Erdős discrepancy problem, Tao's logarithmic version of the Chowla and Elliott conjectures suffices to show that a completely multiplicative function f with $|f(n)| = 1$ and bounded partial sums must correlate with functions of the form $\chi(n) n^{i\alpha}$. Indeed given a completely multiplicative f with bounded partial sums, we can carry out a logarithmic van der Corput argument:

$$\frac{1}{\log N}\sum_{n\le N}\frac{1}{n}\Big|\sum_{h=1}^{H} f(n+h)\Big|^2 \le C,$$

and expanding the above we obtain (in place of (15))

$$\frac{1}{\log N}\Big|\sum_{n\le N}\frac{1}{n} f(n+h_1)\overline{f(n+h_2)}\Big| \ge \frac{1}{2H},$$

for some $h_1 \ne h_2 \le H$. Now invoking Tao's work [20] it follows that f must correlate with (or pretend to be) $\chi(n)n^{i\alpha}$ for a suitable character χ and a real number α.

Finishing the proof.

We are now at the last stage of the proof. Suppose for simplicity that f is a completely multiplicative function taking values ± 1 and with bounded partial sums. As described above, f must correlate with some function of the form $\chi(n)n^{i\alpha}$, and since f is real, one can ensure that $\alpha = 0$ and χ is real as well. Let q denote the conductor of the character χ, and define $\widetilde{\chi}$ to be a completely multiplicative function with $\widetilde{\chi}(p) = \chi(p)$ for all $p \nmid q$, and $\widetilde{\chi}(p) = f(p)$ for $p|q$. Put $f(n) = \widetilde{\chi}(n)g(n)$ for a completely multiplicative function g taking values ± 1. Assume that the correlation of f and χ takes the simplified form

$$\sum_p \frac{1 - f(p)\widetilde{\chi}(p)}{p} = \sum_p \frac{1-g(p)}{p} \le C, \tag{17}$$

for some constant C.

The idea is to decouple the character-like function $\widetilde{\chi}$ from g (which is close to the function 1). Let H be a large integer, and k be such that $2^k > 2H$. Let X be much larger than q^k. Since the partial sums of f are bounded, it follows that

$$\Big|\sum_{n\equiv 0 \mod q^k}\frac{1}{n^{1+1/\log X}}\sum_{h=1}^{H} f(n+h)\Big| \le A\frac{\log X}{q^k},$$

for some constant A. If $n \equiv 0 \mod q^k$ then for all $h \le H$ we see that $(n+h, q^k) = (h, q^k)$ must be a divisor of q^{k-1}, and therefore $f(n+h) = \widetilde{\chi}(n+h)g(n+h) = \widetilde{\chi}(h)g(n+h)$. Using this above, we obtain

$$\Big|\sum_{h=1}^{H}\widetilde{\chi}(h)\sum_{n\equiv 0 \mod q^k}\frac{g(n+h)}{n^{1+1/\log X}}\Big| \le A\frac{\log X}{q^k}. \tag{18}$$

The hypothesis (17) says that g is close to the constant function 1 and, rather as in Wintner's result (12), it is not hard to show that

$$\sum_{n\equiv 0 \mod q^k} \frac{g(n+h)}{n^{1+1/\log X}} \approx \frac{\log X}{q^k}\prod_p \Big(1-\frac{1}{p^{1+1/\log X}}\Big)\Big(1-\frac{g(p)}{p^{1+1/\log X}}\Big)^{-1} \approx \mathfrak{S}\frac{\log X}{q^k},$$

for some constant $\mathfrak{S} > 0$. Roughly, this says that g is equidistributed in residue classes $\mod q^k$, and in establishing this, one uses that $(h, q^k)|q^{k-1}$, and that $g(p) = 1$ (by construction) for all $p|q$. Inserting this in (20) we conclude that

$$\Big|\sum_{h=1}^{H} \widetilde{\chi}(h)\Big| \le B, \tag{19}$$

for some constant B.

Thus, from knowing that the partial sums of f are bounded, we have passed to knowing that the partial sums of the character-like function $\widetilde{\chi}$ are bounded. At this stage, let us simplify our task once more and assume that the modulus q is a prime number. We are now back to the situation of Example 2! Write H in base q as $H = h_0 + h_1 q + \ldots + h_r q^r$, where $0 \le h_j \le q-1$ and $h_r \ge 1$. Then, a small calculation shows

$$\sum_{h=1}^{H} \widetilde{\chi}(h) = \sum_{j=0}^{r} f(q)^j\Big(\sum_{n\le h_j}\chi(n)\Big).$$

Choose $h_j = 0$ if $f(q)^j = -1$ (which happens only if $f(q) = -1$ and j is odd), and $h_j = 1$ if $f(q)^j = 1$ (which happens whenever j is even, and if $f(q) = 1$ then for all j). With this choice the above is at least $\lfloor \log H/(2\log q)\rfloor$, which goes to infinity with H. This contradicts (21), and completes our (oversimplified) proof sketch.

Let H be a large parameter, k a natural number much larger than H, and X a parameter still much larger than q^k. Since the partial sums of f are bounded, for any residue class $a \mod q^k$ we have

$$\sum_{n\equiv a \mod q^k} \frac{1}{n^{1+1/\log X}}\Big(\sum_{h=1}^{H} f(n+h)\Big)^2 \le A^2\frac{\log X}{q^k},$$

for some constant A. Using Cauchy-Schwarz we deduce that

$$\Big(\sum_{n\equiv a \mod q^k} \frac{1}{n^{1+1/\log X}}\sum_{h=1}^{H} f(n+h)\Big)^2 \le 2A\Big(\frac{\log X}{q^k}\Big)^2. \tag{20}$$

Call a residue class $a \mod q^k$ *good* if $(a+h, q^k)|q^{k-1}$ for all $h \le H$. The number of residue classes that are not good is $\le H\sum_{p|q} q^k/p^k \le 10Hq^k/2^k$, so that the vast majority of residue classes are good. If $a \mod q^k$ is good then $\widetilde{\chi}(n+h) = \widetilde{\chi}(a+h)$ for any $n \equiv a \mod q^k$ and all $h \le H$. Therefore, for a good residue class $a \mod q^k$ the estimate (20) gives

$$\Big(\sum_{h=1}^{H} \widetilde{\chi}(a+h) \sum_{n\equiv a \mod q^k} \frac{g(n+h)}{n^{1+1/\log X}}\Big)^2 \le 2A\Big(\frac{\log X}{q^k}\Big)^2.$$

. Thus we conclude that for a suitable constant B, and any good residue class $a \mod q^k$ one has

$$\Big(\sum_{h=1}^{H} \widetilde{\chi}(a+h)\Big)^2 \le B,$$

and so, with $*$ indicating a sum over good residue classes,

$$\sideset{}{^*}\sum_{a \mod q^k} \Big(\sum_{h=1}^{H} \widetilde{\chi}(a+h)\Big)^2 \le Bq^k. \tag{21}$$

If a is good then $(a+h, q^k)$ divides q^{k-1}, and $\widetilde{\chi}(a+h) = \chi((a+h)/d)$. Thus, expanding out (21) we obtain

$$\sum_{d_1,d_2|q^{k-1}} \sum_{h_1,h_2\le H} \sideset{}{^*}\sum_{\substack{a \mod q^k \\ d_j|a+h_j}} \chi\Big(\frac{a+h_1}{d_1}\Big)\chi\Big(\frac{a+h_2}{d_2}\Big) \le Bq^k$$

Since the vast majority of residue classes are good, we may include the bad residue classes also in the sum over a above, incurring an error term of size at most $10d(q^{k-1})^2H^3q^k/2^k$ where $d(q^{k-1})$ denotes the number of divisors of q^{k-1}. This is negligible compared to q^k, since k is assumed to be much larger than H. Therefore

$$\sum_{d_1,d_2|q^{k-1}} \sum_{h_1,h_2\le H} \sum_{\substack{a \mod q^k \\ d_j|a+h_j}} \chi\Big(\frac{a+h_1}{d_1}\Big)\chi\Big(\frac{a+h_2}{d_2}\Big) \le 2Bq^k.$$

Now a little calculation shows that the sum over a cancels to zero unless $d_1 = d_2$, so that

$$\sum_{d|q^{k-1}} \sum_{a \mod q^k} \Big(\sum_{\substack{h\le H \\ d|a+h}} \chi\Big(\frac{a+h}{d}\Big)\Big)^2 \le 2Bq^k. \tag{22}$$

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STATISTICAL PROOF?
THE PROBLEM OF IRREPRODUCIBILITY.

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Abstract. Data currently generated in the fields of ecology, medicine, climatology and neuroscience often contain tens of thousands of measured variables. Statistical analyses can result in publications whose results are irreproducible.

The field of modern statistics has had to revisit the standard hypothesis testing paradigm to accommodate modern high throughput settings. A first step was correction for multiplicity in the number of possible variables selected as significant using multiple hypotheses correction and FDR control (Benjamini, Hochberg, 1995). FDR adjustments do not solve the problem of double dipping the data and recent work develops a field known as post-selection inference to try to enable inference when the same data is used both to choose and evaluate models.

It remains that the complexity of software and flexibility of choices in tuning parameters can bias the output towards inflation of significant results; neuroscientists recently revisited the problem and found that many fMRI studies have resulted in false positives.

Unfortunately all formal correction methods are tailored for specific settings and do not take into account the flexibility available to today's statisticians. A constructive way forward is to be transparent about the analyses performed, separate the exploratory and confirmatory phases of the analyses and provide open access code such as that provided by Bioconductor; this will result in both enhanced reproducibility and replicability.

1. Introduction

Statistics is the main inferential tool used today in science and medicine. It enables us to draw conclusions or make decisions based on noisy data collected under constraints of cost and time.

Classical statistical hypothesis testing originated with Fisher, Neyman and Pearson and was founded on the premise that starting with one hypothesis H_0, then designing an appropriate experiment and collecting data $\mathcal{X}$, one could compute the probability that the data observed could occur under that hypothesis: if the probability $P(\mathcal{X}|H_0)$ was small, say smaller than $\alpha = 0.05$, one would then proceed to reject the plausibility of H_0, we call this a discovery.

However, modern experiments produce high throughput data with tens of thousands of measurements or even complete images containing hundreds of thousands of voxels. These lead to multiple hypotheses and variables on which statisticians

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have to make exponentially many preprocessing choices. The use and abuse of statistics in these high-throughput settings has led to an outpouring of negative press about the scientific method and p-values in particular [47].

I will argue that new statistical, mathematical and computational tools, and scientific transparency, enable us to clean up our act. We should not be throwing out the baby with the bathwater.

This review starts with a few examples that will help focus our attention on the problem of multiplicity and the current crisis in reproducibility. I will then explain what the essence of statistical testing is, explaining what a p-value and nul hypothesis are in a simple example. We will explain how some of the methods for controlling false discovery rates work. We give a few examples of applying new ideas to the problem of multiplicity in the context of variable selection which attempts to adjust for some of the flexibilities of modern statistical methods. The last section will give strategies that improve reproducibility such as careful design and computational strategies and the idea that transparency can act as surrogate for proof.

2. Some examples

Here is a simple example. Suppose that a specific site in a protein is tested to see if it is an epitope (i.e. creates an allergic reaction) using an ELISA test [2]. The outcome will be 1 or 0, 1 indicating a reaction. Due to the nature of the test, sometimes a 1 occurs by chance, it is known that this happens with probability $\frac{1}{100}$. Let the null hypothesis H_0 be 'the site does not create a reaction'. Suppose 50 people are tested and there are four 1's reported. If the null hypothesis is true, the chance of observing four or more 1's is well approximated by a Poisson with probability $\frac{1}{2}$ (Law of small numbers), more generally the Poisson distribution with parameters λ is the probability measure on $\{1, 2, 3 \ldots\}$ which has $P_\lambda(j) = e^{-\lambda}\lambda^j/j!$. Thus in our data

$$P\{4 \text{ or more } |H_0\} = \sum_{j=4}^{\infty} e^{-1/2}(1/2)^j/j! = 0.00175$$

This seems an unlikely event and standard practice rejects the null hypothesis with a p-value equal to 0.00175. The above tale is the way things are supposed to be done. In reality, the immunologist who gave us the data might well have tried 60 potential positions and reported the site with the largest numbers of ones.

However, if the design was actually not quite as straightforward and we find out that the immunologists had actually started by exploring 60 potential positions whose cumulative scores for the 50 patients are represented in Figure 1, then the computations and conclusion change. Does the evidence still point to a possible epitope?

The computation then has to factor in that it wasn't a random position that was chosen, but the best out of 60. We order the data values $x_1, x_2, \ldots, x_{60}$ renaming them $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(60)}$ so that $x_{(1)}$ denotes the smallest and $x_{(60)}$ the largest of the counts over the 60 positions.

[2]Enzyme-Linked Immunosorbent asSAy, see https://en.wikipedia.org/wiki/ELISA

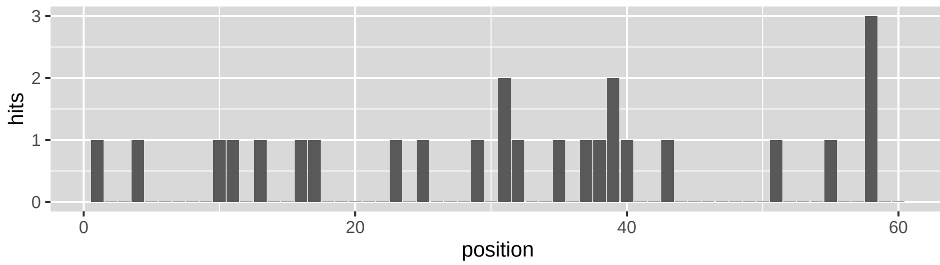

FIGURE 1. Output of an ELISA array for 50 patients in the 60 positions.

The maximum value being as large as 4 is the complementary event of having all 60 independent counts be smaller or equal to 3 with the Poisson of rate $\lambda = 0.5$.

$$\begin{aligned} P(x_{(60)} \geq 4) &= 1 - P(x_{(60)} \leq 3) = 1 - \prod_{i=1}^{60} P(x_i \leq 3) \\ &= 1 - \left(\sum_{k=0}^{3} \frac{e^{-\lambda}\lambda^k}{k!} \right)^{60} = 1 - 0.900 = 0.10 \end{aligned}$$

The p-value is no longer small enough to justify rejecting the null hypothesis of the positions not being an epitope. We see that flexibility in the choice of the position changed the conclusion of the test.

We can follow this simple computation with an interesting mathematical aside. An elegant part of probability deals analytically with the maximum of independent random variables. If our original n measurements followed a Gaussian distribution and $\max_n = X_{(n)}$ is their maximum, the theory says:

$$P\{X_{(n)} \leq \sqrt{2\log n - \log\log n + 2x}\} \sim e^{\frac{-e^{-x}}{\sqrt{2\pi}}}, \qquad \text{for any fixed } x, -\infty < x < \infty$$

In practice,we would find x so the right hand side is 0.99 and use replace this x on the left hand side to fixe the confidence bound. The right hand side is called the Gumbel or extreme value distribution. It has a universal quality; many different underlying distributions for X_i have the same limit.

For the maximum of independent Poisson(λ) random variables, it would be nice if we could find a_n and b_n so that when n is large

$$P\{\frac{X_{(n)} - a_n}{b_n} \leq x\} \sim e^{-e^{-x}}$$

Suprisingly here, the discreteness really matters. It can be proved that there is no possible such a_n, b_n for Poisson or even rounded Normal random quanitites. Number theory issues have to be dealt with, however there are still useful approximations available, see [12] [18]. Of course, today it is straightforward to use Monte Carlo to approximate distributions, although tail probabilities and the calculation of probabilities of rare events still pose challenging problems.

The advent of modern nonparametric visualization and computational tools enable flexibility in the choice of endpoints studied and multiple opportunities for Pareidolia[3], these are at the core of the problems facing statistical inference today.

[3]https://en.wikipedia.org/wiki/Pareidolia

Unfortunately the type of extreme value theory shown above is not sufficient to solve the multiplicities involved.

2.1. **Most research findings are false?** The last few years have seen the questioning of the biomedical research as had been practised for over forty years. Medical research is usually published if the p-values showing an improvement or the significance of a result smaller than $\alpha = 0.05$ or sometimes $\alpha = 0.01$. However, many authors have questioned the validity and reproducibility of results that ignore the file drawer effect [61]: a natural bias that occurs when experiments which do not show statistically significant differences cannot be published. John Ioannidis [35] went so far as to challenge that *"most published research findings are false"*, his work has had strong repercussions in biomedical world where the pressure to publish has economic underpinnings. His paper focuses on how application of the hypothesis testing framework could result in a preponderance of false positive results in the medical literature. The key argument is analogous to population based disease screening (think TB test). A very sensitive and specific diagnostic may still have a low positive predictive value if the disease is not prevalent in the population. His premise is that significant hypotheses are rare and he takes a prevalence of possible significant results at 1%, meaning that the null hypothesis is true for 99% of the hypotheses being considered, and 1% of the tested hypotheses are indeed false and should be rejected. Statistical power measures how likely a test is to reject a test when it should, i.e. the true positive rate [4]. If scientists test 1,000 hypotheses with a statistical power of 80%. Then they will be 1, 000 1% 80% = 8 true alternative hypotheses correctly detected. Even though the false positive rate is much lower, the prevalence of null hypotheses is much higher. With a false positive error rate of $\alpha = 0.05$, we expect $1000 \times 0.99 \times 0.05 = 49.5$ null hypotheses will be incorrectly rejected (we'll call thse false discoveries). So 49.5/(57.5)= 86% of rejected hypotheses will actually be null. Assuming selective reporting of positive results, a lower prevalence of true alternatives or other sources of bias in the publication process could make this estimate even worse.

These numbers were challenged by a subsequent analysis based on collecting all the p-values in more than 5,000 papers in medical journals [37], their model for p-values uses a mixture model and leads them to a less dramatic false discovery rate closer to 20%, however this actually depends on other parameters such as the power of the studies. Many studies today suffer from being underpowered, there is a real lack of rigor in experimental design, this is a real problem [15] that we will not delve into here. There are some things that can be done to adjust for some of the flexibility and we will give some details in the next section. But first a little reminder that statistics is a field with its own subtleties where even the most brilliant mathematics can go astray.

2.2. **Orion and the star constellations.** David Mumford [53] suggested in a paper entitled *"Intelligent design found in the sky with $p < 0.001$"* that the positioning of the stars in the Orion constellation were so particular that only intelligent design could explain them. However in his calculations, he starts by asking what the probability is that the stars of Orion are so aligned. This argument suffers from the blade of grass paradox, as put by Diaconis (NYtimes, 1990): *If you were*

[4]the statistical power of a test is often written $1 - \beta = 1 - P$(do not reject $|H_0$ is false), where β denotes the false negative rate

to stand in a field and reach down to touch a blade of grass, there are millions of grass blades that you might touch. But you will, in fact, touch one of them. The a priori fact that the blade you touch will be any particular one has an extremely tiny probability, but such an occurrence must take place if you are going to touch a blade of grass.[5] Why not ask what the chances of an equilateral triangle or other formation are?

2.3. **The Bible Codes.** Witztum, Rips and Rosenberg [82] used a statistical test to show surprising proximities between equidistant letter sequences (ELS) of names of famous rabbis and their known dates of birth and death. This work was endorsed by David Kazhdan, Joseph Bernstein, Hillel Furstenberg, Ilya Piatestski-Shapiro who wrote in the forward to the book:
The present work represents serious research carried out by serious investigators.

McKay, Bar-Natan, Bar-Hillel and Kalai [51] showed the existence of a certain flexibility in the ELS experiment and give plenty of evidence for biased data-selection. One thing to keep in mind is that this flexibility does not result in the addition of small effects, but as in the maximum problem in the previous section: multiplicative effects, and multiplying many numbers smaller than one can result in very small probabilities indeed.

3. Adjustments for multiple hypotheses

Formalizing the problem needs some more definitions and terminology. We have encountered the false positive rate α, sometimes called type I error. The type II error or false negative rate is the probability of not rejecting a hypothesis that is in fact false and is denoted as β. To take into account the high-throughput nature of today's data, suppose that m different potential hypothese $M_1, M_2, \ldots, M_m$ are being tested at once, this is also called the simultaneous inference problem.

3.1. **Correction of p-values through control of the FDR.** Start with a simple but important mathematical fact: under the null distribution the p-values should be uniformly distributed between 0 and 1. Suppose that in fact m_0 of the hypotheses are truly null. Some of these will lead to mistakenly rejecting the null hypotheses. Suppose we actually reject the hypothesis R times and call these the discoveries. V among them are actually not true discoveries but correspond to random uniform p-values smaller than the cutoff α we specified (α, the false positive rate is often fixed at 0.05 or 0.01). Replacing the cutoff α by α/m provides the what is known as the Bonferroni correction for multiple testing. This gives correct bounds, but is very conservative often leading to the rejection of too few hypotheses.

The FDR or false discovery rate is the expected value of V/R, the family wise error rate (FWER) is the probability that V/R is non zero. Benjamini and Hochberg [5] remarked that it is "desirable to control the expected proportion of errors among the rejected hypotheses" and defined a procedure similar to [64] that controls the FDR. It is important to notice that the FDR is an expected value and is not associated to just one hypotheses but to the ensemble of m hypotheses being tested in the simultaneous inference. Here is a step by step description of the

[5]A similar argument by Einstein was related by Wigner in his autobiography: *Life is finite. Time is infinite. The probability that I am alive today is zero. In spite of this, I am now alive. Now how is that?* None of his students had an answer. After a pause, Einstein said, *Well, after the fact, one should not ask for probabilities..*

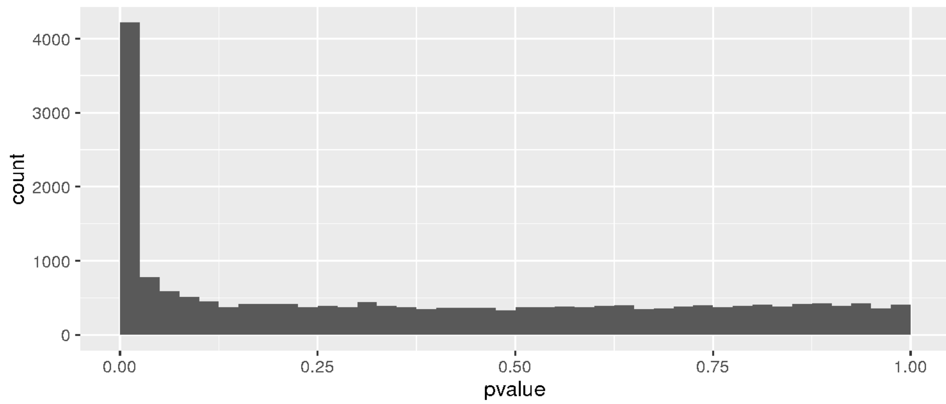

FIGURE 2. A histogram of p-values from a gene expression experiment involving more than 20,000 genes measured through counting RNA-sequencing reads [32].

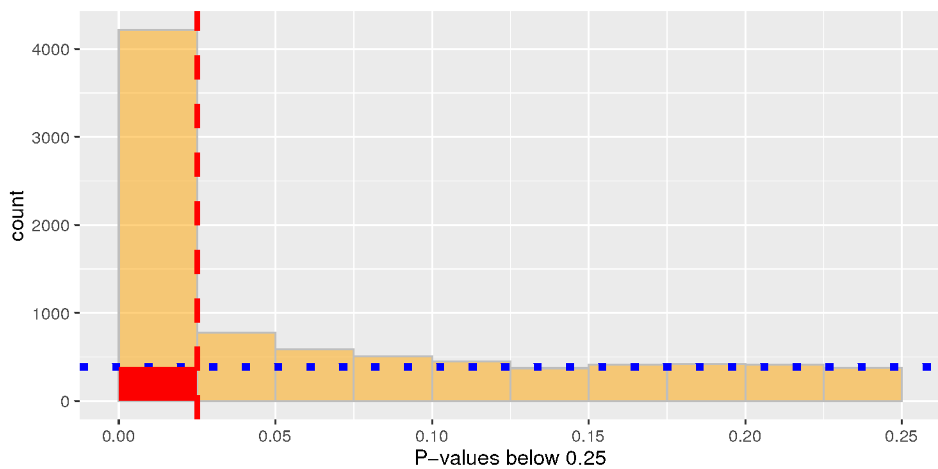

FIGURE 3. The darkened area in the lower left side represents a possible estimate of the false discovery rate from one instance of the p-value histogram (only the left side is shown to maximize resolution), with the false positive rate set at $\alpha = 0.025$. The dotted line shows the estimated p-value density under the null. This is a closeup magnification of the data shown in Figure2 (see [32] for a complete description).

Benjamini and Hochberg (BH) method for controlling the FDR:
- Fix a level γ to which we want the FDR to be proportional to.
- Sort the observed p-values, $p_{(1)} \leq p_{(2)} \leq \ldots \leq p_{(m)}$ and compare $p_{(k)}$ to $k \cdot \gamma/m$.
- Let $r = \max\{p_{(k)} \leq k{\cdot}\gamma/m\}$ reject the first r hypotheses hypotheses $M_{(1)}, \ldots, M_{(r)}$.

This is illustrated in Figure 4 which shows the ranked p-values for the lowest 6,000 out of 20,000 genes of an RNA-seq experiment (see details in [32]) as they compare to the line $k \cdot \gamma/m$ with γ taken to be 0.10.

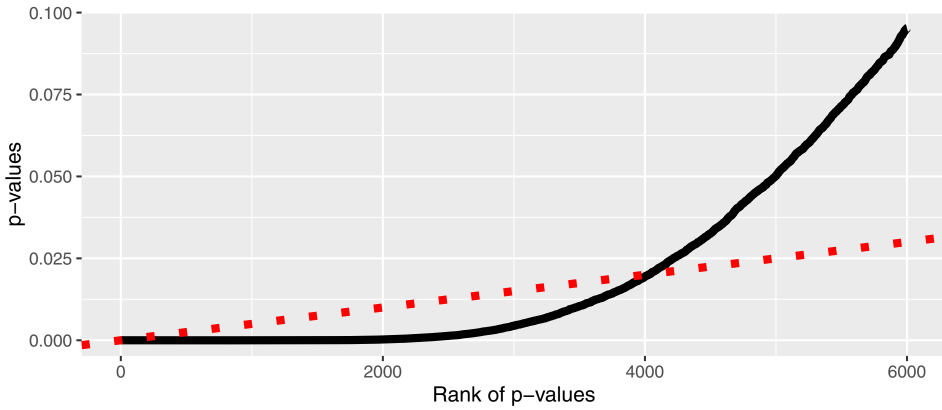

FIGURE 4. The Benjamini Hochberg procedure applied to the RNA-seq data comparing airway smooth muscle cell lines as described in [32] and whose p-values are plotted in Figure2 suggests taking the first 4,100 p-values as significant.

This procedure was designed to work for independent tests. Later theoretical work shows that it works even if the test statistics are dependent. Under independence of the hypotheses this procedure leads to $FDR = m_0 \cdot \gamma/m$. Under positive dependence $FDR = m_0 \cdot \gamma/m$ [7] and general dependencies $FDR \leq (1 + \frac{1}{2} + \ldots + \frac{1}{m}) \cdot m_0 \cdot \gamma/m$.

Mixture model for testing. The advent and importance of adjustments for multiple hypotheses testing was motivated in the context of microarrays, where tens of thousands of genes are tested in two different conditions such as normal and cancer cells. Scientists were eager to conclude that they were able to find significant differences in gene expression. Unfortunately many of these discoveries were later shown to be impossible to replicate [36].

One of the approaches statisticians [54, 24, 69, 68] developed to address the multiple testing correction problem in this context was to introduce a mixture model. Careful inspection of Figure 2 that the p-values suggest such an approach as natural. It seems that a proportion of the p-values are clustered at zero, whereas a remaining proportion (π_0) are uniformly spread throughout the $[0, 1]$ interval.

These models are called hierarchical, as there is a hidden layer above the observed data, a latent variable is used to distinguish the truly differentially expressed genes from the others.

Suppose we write the statistic measured in the m experiments $T = (T_1, T_2, \ldots, T_m)$ and we assume our hypotheses are $M_k = H_0 = 0$ if the kth hypothesis is truly null, in which case T_k will come from a density f_0, we write $T_k \sim f_0$. Otherwise, $M_k = 1$ and $T_k \sim f_1$. We assume further the 'hierarchical' parameter π_0 such that M_k are Bernoulli random variables with probability $1 - \pi_0$.
We can write that the distribution of T_k's:

$$T_k|M_k \text{ has the density } f(t) = \pi_0 f_0(t) + (1 - \pi_0) f_1(t).$$

If we consider the T_k's to be the p-values, $f_0(p) = 1$ and we have

$$f(p) = \pi_0 + (1 - \pi_0) f_1(p) \text{ (see Storey [68] or Efron [21] for details).}$$

What we really want to know is the probability that a hypothesis is true $P(M_k|T_k = t)$. In the traditional frequentist context this would be meaningless; however a Bayesian can use the mixture model and Bayes theorem to compute this probability:

$$P(M_k = 0|T_k = t) = \frac{\pi_0 f_0(t)}{f(t)} = \frac{\pi_0 f_0(t)}{\pi_0 f_0(t) + (1-\pi_0) f_1(t)}.$$

Now controlling the positive FDR (pFDR) defined as

$$E(\frac{V}{R}|R > 0) = Pr(H_k = 0|R_k = 1) = \frac{EV}{ER}(\text{also denoted Fdr by Efron et al. [24]})$$

becomes a Bayesian classification problem. This is explained by Storey, 2003 [68] showing that in fact we are trying to decide whether a hypothesis belongs to a the null group or not.

For $R > 0, V$ is distributed as a Binomial on R trials with probability Fdr. By defining $\{\Gamma_\alpha\}_{0\leq\alpha\leq 1}$ a nested set of significant regions such that $\{\Gamma_{\alpha'} \subset \Gamma_\alpha$ for $\alpha' < \alpha$, then Storey (2003)[68, Theorem 1] shows that the positive FDR is

$$pFDR(\Gamma_\alpha) = Pr(M_k = 0|T_k \in \Gamma_\alpha) = \int_{\Gamma_\alpha} Pr(M_k = 0|T_k = t) f(t|t \in \Gamma_\alpha) dt$$

The significance regions $\Gamma_\alpha = \left\{t : \frac{\pi_0 f_0(t)}{f(t)} \leq \alpha\right\}$ minimize the Bayes Error of classifying $\{M_k\}$ using the standard $R_k(T_k, \delta(q))$ classification procedure for which $Fdr = P(M_k = 0|R_k = 1) = q$.

Of all rules $R_k(T_k, \delta)$ with $Fdr = q$, the Bayes classifier has maximum power to make discoveries $P(R_k = 1|M_k)$ and minimum false negative rate: $P(M_k = 1|R_k = 0)$, see [21] for a book long treatment.

In practice, several implementations in R are now in widespread use to control the pFDR(Fdr), among them:

- `locfdr` package estimates π_0, f_0, f_1 and computes the Fdr of the rejection rule $R_k = \mathbb{I}_{z\leq Z}$.
- `qvalue` package in R: computes the pFDR of the rejection rule: $R_k = \mathbb{I}_{p_k \leq p}$.

$$P(H_k = 0|p_k \leq p) = \frac{H_k = 0, p_k \leq p}{P(p_k \leq p)} = \frac{P(p_k \leq p|H_k = 0)\pi_0}{P(p_k \leq p)}$$

$$\widehat{qvalue} = \frac{p \times \hat{\pi}_0}{\#\{k : P_k \leq p\}/m}, \qquad \hat{\pi}_0 = 2 \times \#\{k : 0.5 \leq P_k\}$$

Note that this is how the blue line in Figure 3 was drawn.

3.2. **There are more to the data than p-values.** The previous methods all depended on the correction of the p-values taken alone, however recent work has shown that improvements can be made by using all the data available. Originally one has the variance of the test statistics and even useful covariates that can inform the classification into true and false positives. Stephens [66] uses unimodality of the underlying effect and estimates of the standard errors as well as the effect sizes, using more of the information improves the power of his procedure. The estimation of the unimodal distribution only involves solving a convex optimization problem- and enables more accurate inferences provided that the assumptions hold. His method is available in the R package `ashr`. This method uses enriched information facilitates the estimation of actual effect sizes, providing credible regions for each effect in addition to measures of significance.

Ignatiadis et al. [34] propose a weighting scheme that also enriches the p-value information using covariates independent of the p-values under the null hypothesis. This method also increases the statistical power $(1-\beta)$ to discover associations and is implemented in the `IHW` Bioconductor package.

4. Model selection

There is also need to provide inference (p-values or confidence statements) in a slightly different framework than the simultaneous inference and multiple testing explored above. Modern statistical applications such as finding signal in fMRI images or predicting a response variable with nonparametric regression require marginal inferences on multiple parameters selected after a preliminary exploration of the data. The problem is common: data such as $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ is collected. A preliminary plot shows the y_i is approximately a quadratic function of x_i. Now, a statistical test of the quadratic coefficient is carried out. How should the preliminary plotting be accounted for?

This is called post selective inference, a subject that has seen a healthy development in the last five years. Post hoc analyses were studied by Tukey and Scheffe in the 1950's to deal with inference in Analysis of Variance (ANOVA) where one significant contrast was chosen from the data; a complete review of methods anterior to 1995 can be found in [63]. However the era of high dimensional data is upon us and *double dipping* [41] the same data set at the selection and evaluation steps invalidates standard analytics: the statistics are not independent of the selection criteria under the null hypothesis. Several groups have been investigating rigorous approaches to tackle this issue.

4.1. Post-selective inference after model selection for standard linear models.

Classification and regression were traditionally set in a fixed model context. A **response** y is the variable statisticians attempt to predict from a given set of variables. The model can be written: $y = f(x_1, x_2, \ldots, x_p) + \epsilon$ where ϵ is the noise, often supposed to have expectation 0 and be modeled as an independent identically distributed random variable. However, the growing size of p and in particular the nonparametric nature of the cases where $p > n$ have introduced new procedures (for a full review see [30] for instance) where a subset of the explanatory variables is chosen to ensure a sparser, regularized model. Thus for instance forward stepwise regression is a sequential procedure that steps through the possible predictors and aims to provide a parsimonious set of variables. LASSO regression imposes a penalty on non zero coefficients in a regression and ensures that the model has a lower variance and less overfitting than a standard least squares one would. Adaptive regression models are stochastic and as pointed out by Berk and co-authors [8] pose serious challenges to the classical inferential paradigm:

> *"Posed so starkly, the problems with statistical inference after variable selection may well seem insurmountable. At a minimum, one would expect technical solutions to be possible only when a formal selection algorithm is (1) well-specified (1a) in advance and (1b) covering all eventualities, (2) strictly adhered to in the course of data analysis and (3) not improved on by informal and post-hoc elements. It may, however, be unrealistic to expect this level of rigor in most data analysis contexts, with the exception of well-conducted*

> *clinical trials. The real challenge is therefore to devise statistical inference that is valid following any type of variable selection, be it formal, informal, post hoc or a combination thereof."*

The authors propose a valid Post Selective Inference (PoSI) with family-wise error guarantees that account for all types of variable selection. This involves ignoring the variables whose coefficients in the model are estimated at zero. Suppose we want to use the equation merely to describe association between predictors and response variables. Here the predictors are associated to each other and to the response. The model $\hat{M}$ is random and depends on the observed response y. The consequences of their theory of PoSI in the linear model case is that by making the standard confidence bound (the ones often used follow a $t_{1-\alpha/2,r}$ distribution), using a larger constant $K(X_{n\times p}, M, \alpha, r)$ than that provided by the standard t one can provide simultaneity protection for up to $p \cdot 2^{p-1}$ parameters $\beta_j \cdot M$, K depends strongly on the predictor matrix X as the asymptotic bound for $K(X_{n\times p}, M, \alpha, r)$ with $d = rank(X)$ ranges between the minimum of $\sqrt{2\log d}$ achieved for orthogonal designs on the one hand, and $\sqrt{d}$ on the other hand. This wide asymptotic range suggests that computation is critical for problems with large numbers of predictors. In the classical case d = p their current computational methods are feasible only up to about $p \simeq 20$. In fact the key in the above discussion is to declare that M is random and present an honest way to account for randomness in both the hypotheses: such a model is available through the Bayesian paradigm described in the next section.

4.2. **Bayesian Selective Inference.** In practical situations, statisticians start by finding interesting parameters: a random process, then we want to provide inferences for these selected parameters. In fact, the attentive reader will realize that there is a real challenge posed in providing a frequentist interpretation of post selective model selection. The randomness of the model inhibits a long term frequency interpretation of statements such as those of the following [8, equation 4.6]

$$P\left\{\beta_{j,\hat{M}} \in CI_{j,\hat{M}}(K), \ \ \forall \hat{M}\right\}$$

This can be simply overcome by doing what Storey did for multiple testing and posing the problem in a Bayesian perspective. We can compute the probabilities conditioning on the coefficients being non zero.

Yeketuli [83] developed such a framework for accounting for truncation of the data in the selection process. Here we present a very elegant example that captures a subtlety due to the conditional nature of the inferences and truncation that occurs.

4.2.1. *Example from* [83]. Suppose we wish to predict a student's true academic ability from their observed/tested ability but only for the students admitted to college. We use the following toy model.

- The students have a true (unknown) academic ability $\theta_i \sim \mathcal{N}(0,1)$, notice the Bayesian can take a true parameter to have a distribution.
- Observed academic ability in high school $Y_i \sim \mathcal{N}(\theta_i, 1)$.
- Only students with $Y_i > 0$ are admitted to college.

We want $f_S(\theta_i, y_i)$ to compute the prediction error

$$\int_{\theta_i=-\infty}^{\infty}\int_{y_i=0}^{\infty} f_S(\theta_i, y_i)\cdots(\theta_i - \delta(y_i))^2 dy_i d\theta_i$$

For the college professor predicting θ for a student in their class, the joint distribution (θ, Y) of a random college student is found by by generating (θ, y) for a random high school student and selecting (θ, y) only if $0 < y$. Thus the joint distribution is:

$$f_S(\theta, y) \propto \frac{\exp(-\frac{\theta^2}{2})\exp(-\frac{(\theta-y)^2}{2})}{P(Y>0)} \propto \exp(-\frac{\frac{(\theta-y)^2}{2}}{2\times\frac{1}{2}})$$

Now for the high school counselor who is assigned to only counsel students whose (unknown) ability would allow them to attend college. The distribution seen comes from $\theta_i \sim \mathcal{N}(0,1)$ the Y_i used to predict θ_i is drawn from $\mathcal{N}(\theta, 1)$ and truncated by $Y_i > 0$. The joint density becomes:

$$f_S(\theta, y) \propto \frac{\exp(-\frac{\theta^2}{2})\exp(-\frac{(\theta-y)^2}{2})}{P(Y>0|\theta)}$$

We see the difference is subtle, but we can distinguish the two densities as shown in Figure 5; in the college population case we divide by $P(Y > 0)$ which will be half the density, whereas in the high school population the probability $P(Y > 0|\theta)$ cannot be computed in closed form but we see this decreases in θ.

Suppose θ is the parameter, Y is the data and Ω is the data sample space. $\pi(\theta)$ is the prior distribution for the parameter of interest and and $f(y|\theta)$ is the likelihood function. The multiple parameters, for which inference may or may not be provided, are actually multiple functions of $\theta : h_1(\theta), h_2(\theta)$, For each $h_i(\theta)$ there is a given subset $S_\Omega^i \subset \Omega$ such that inference is provided for $h_i(\theta)$ only if $y \in S_\Omega^i$ is observed.

Yeketuli's formulation enables a clear description of selection-adjusted Bayesian inference, with $\pi_S(\theta)$ a selection-adjusted prior distribution and as the selection adjusted likelihood he uses the truncated distribution of $Y|\theta$:

$$f_S(y|\theta) = \mathbb{I}_{S_\Omega} \cdot f(y|\theta)/P(Y \in S_\Omega|\theta)$$

Then the Bayes rules are based on the selection-adjusted posterior distribution

$$\pi_S(\theta|y) \propto \pi_S(\theta) \cdot f_S(y|\theta)$$

The authors of [55] have recently also harnessed this approach in an application to variable selection. They rely on an approximation to the full truncated likelihood and can approximate the maximum-likelihood estimate as the Maximum A Posteriori (MAP) estimate corresponding to a constant prior.

4.3. **Postselective inference using a geometric approach.** Jonathan Taylor, Robert Tibshirani and collaborators [77, 74, 71, 26, 22, 49, 78, 76, 29, 80, 42, 73, 75] have approached the same problem of inference after variable selection through a geometric approach.

A very readable account appears in PNAS [74] where the authors take the example of deciding which mutations occurring in HIV strain DNA sequence are significant predictors of drug resistance [60].

Using a standard linear regression model, with response $Y \in \mathbb{R}^n$ and predictors X_j, $j = 1, \ldots, p$.

$$\hat{Y} = \mathbf{X}\beta, \qquad Y \sim N_n(\theta,\ \Sigma).$$

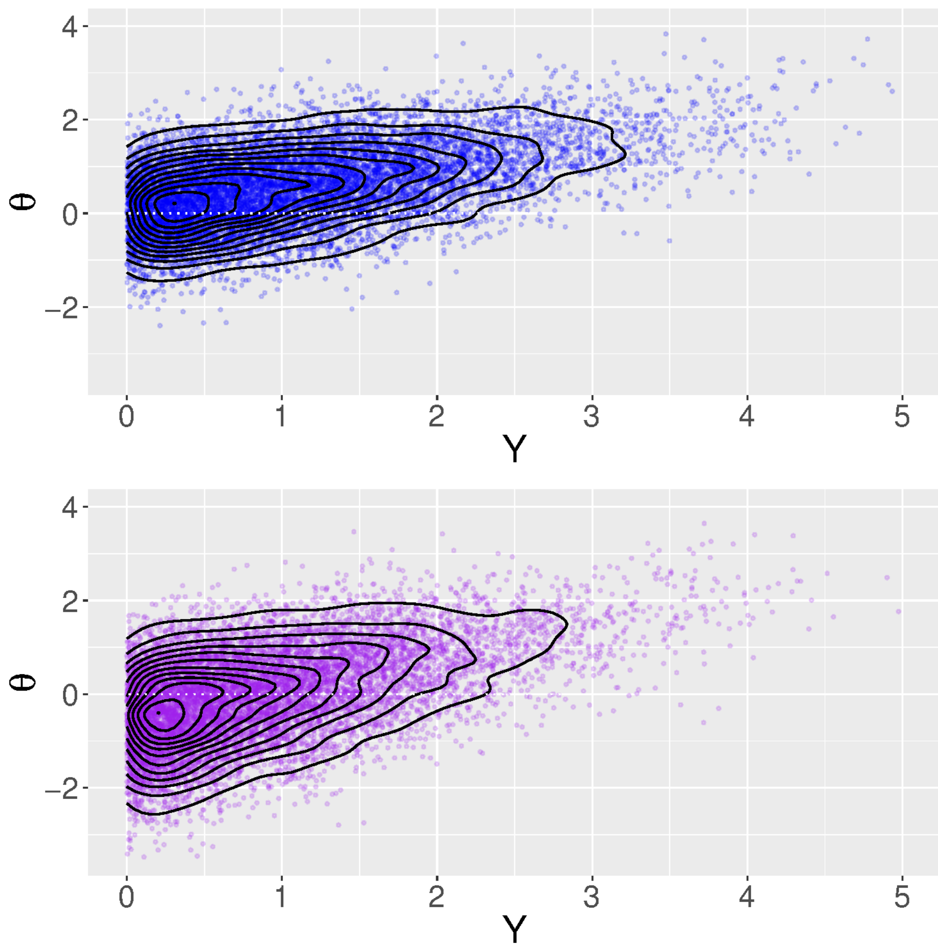

FIGURE 5. Simulation of high school students according to the two different sampling schemes: the college students on the top are drawn conditional on having $Y > 0$, and the high school students in lower plot whose Ys come from a truncated Gaussian. We have used exactly the same parameters for the estimation of the density contours in each case. There are 12 bins and we can see that the lower plot has its center below the dotted white horizontal axis, whereas in the upper plot it is above.

The standard model estimates the parameters by computing

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \; ||Y - X\beta||_2^2 \tag{4.1}$$

To obtain a more interpretable and robust model, we want to take a subset $M \subset \{1, \ldots, p\}$ of the predictors. Each subset M leads to a different model corresponding to the assumption $\hat{Y} = \mathbf{X}_M \beta^M$, where $\mathbf{X}_M$ denotes the matrix consisting of columns X_j for $j \in M$. Then, it is customary to report tests of $H_{0,j}^M : \beta_j^M = 0$ for each coefficient in the model.

We can assume that the selection has the effect of partitioning the sample space into polyhedral (or convex) sets. Jonathan Taylor and co-authors [42] prove the following important result.

Lemma 4.1. *Suppose that we observe $y \sim \mathcal{N}(\theta, 1)$, for any vector η, the condition that y is in the polyhedron$\{Ay \leq b\}$ can be rewritten*

$$\{Ay \leq b\} = \{\mathcal{V}^{\ell} \leq \eta^T y \leq \mathcal{V}^u(z), \mathcal{V}^0(z) \geq 0\}$$

$$\begin{aligned}
\textit{Where } z &\doteq (\mathbb{I} - c\eta^T)y \qquad c \doteq \Sigma\eta(\eta^T\Sigma\eta)^{-1} \\
\mathcal{V}^{\ell}(z) &\doteq \max_{j:(Ac)_j<0} \frac{b_j - (Az)_j}{(Ac)_j} \\
\mathcal{V}^{u}(z) &\doteq \min_{j:(Ac)_j>0} \frac{b_j - (Az)_j}{(Ac)_j} \\
\mathcal{V}^{0}(z) &\doteq \min_{j:(Ac)_j=0} b_j - (Az)_j
\end{aligned}$$

this says that the distribution of $\eta^T y$ conditional on the being in the polyhedron is:

$$\mathcal{L}(\eta^T y | Ay \leq b) = \mathcal{L}(\eta^T y | \{\mathcal{V}^{\ell}(z) \leq \eta^T y \leq \mathcal{V}^u(z), \mathcal{V}^0(z) \geq 0\})$$

$\mathcal{V}^{\ell}(z), \mathcal{V}^u(z), \mathcal{V}^0(z)$ are independent of $\eta^T y$, thus the conditional distribution

$\mathcal{L}(\eta^T y | \{A(m)y \leq b(m), z = z_0\})$ is a truncated Normal.

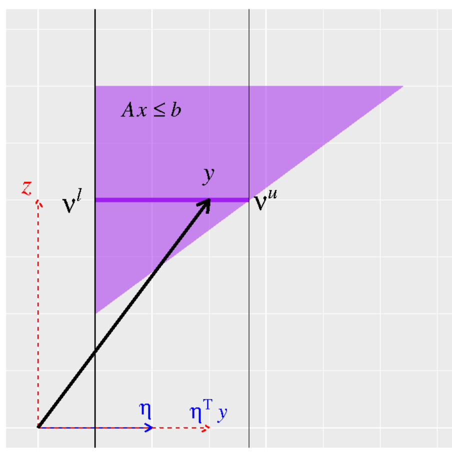

FIGURE 6. The polyhedral lemma, in the special case where $\Sigma = 1$. The bounds represented by the thin vertical lines define the interval of truncation $[V^{\ell}(z), V^u(z)]$ (in dark) to which the Normal is constrained.

This means that by using the inverse cumulative distribution transform, one can find a statistic $F^z(\nu^T\theta)$ whose distribution is uniform enabling inferences on a linear contrast $\nu^T\theta$. A little more work is necessary because in fact several signs are possible and the inferences conditional on the selected model (with the signs s

fixed) being chosen as m with sign s is written $\{M(y) = m, s\}$. The overall inference is then made using a conditional whose support is a series of disjoint intervals, for ample details see [42].

4.4. **Postselective inference using simulated data: the knockoff method.** Another approach to variable selection is to simulate data which are unrelated to the response variable Y but that have the same correlations within the predictors. This has been proposed [3] as the knockoff filter. Multiplying spurious variables which we know should not be entered in the model provides a comparison set of nulls, the knockoffs, to which the other variables can be compared. One has to make sure that the model does not contain the fake variable before the true variable associated to it. This is a promising tool which is generating a healthy development of followup work [39, 4].

4.5. **Blinding in particle physics.** Physicists have advocated a clever method to prevent unconscious bias in the choice of variables, tuning parameters and data transformations called blinding [40]. As in the construction of double blind clinical studies, in the presence of different groups of observations, one can hide the labels of the observations and even assign change the signs to certain variables. These changes are recorded and only revealed after the data analyses have been completed. The method is tuned to physics experiments, an open research probelme is to adapt it to other scenarios.

5. Best practices

5.1. **Reproducible data analyses; replicable scientific experimentation.** It may seem that given the long list of defensive caveats, a quote and a message of hope from Fred Mosteller is in order:

It is easy to lie with statistics, but a whole lot easier without them.

Statistics still has a huge role to play in validating scientific discoveries. However, a clear path needs to be traced between all the pitfalls.

Let's start by clarifying the vocabulary somewhat. In [44], the authors make a clear and useful difference between **reproducibility** and **replicability**; both are desirable in science but pertain to different steps in the scientific process. A piece of work involving statistical code and data is called reproducible if another scientist can take their code and data and reproduce the figures and statistics published in their article. A study is replicable if another scientist can redo the experiment in their laboratory according to the same experimental design, collecting different data, do a statistical analysis similar to the one done by the previous group and come to the same conclusion.

For important medical decisions, it is crucial that data and the code be published and be openly accessible, as shown in the case [2], reproducing the analyses can uncover serious mistakes and save lives and money.

Providing the complete computational workflow is a very important insurance policy against over-tuning the parameters and preprocessing so that a 'clean' outcome and narrative can be published. As shown in [16], if one counts the number of possible analyses on the same data, allowing for the choice of up to 9 outliers, different transformations of the data, choice from 40 different possible distances, 5 different ordination methods, the result is more than 200 million possibilities, no multiple hypotheses correction can protect the user. The only feasible strategy is to allow the reader to run the code and make changes to check the robustness of the conclusions. Robustness is an important component to consider and a large literature on robustness to assumptions, distributional, outliers and statistical functionals exists. It seems that the lessons learned by these studies are not always understood by practionners and recent studies by neuroscients [25] on the inflation of false positive rates concluded that the scientists were using parametric Gaussian assumptions without checking their validity. The same problem drives Taleb's black swan argument [70].

The first problem is that scientists often use black box procedures they don't understand. This can be fixed by encouraging open access to all the code and data used in the analyses and providing educational resoruces. The second problem, sensitivity to unmet assumptions, can be addressed by using nonparametric robust permutation tests instead of parametric ones.

5.2. **A clear division between Exploratory and Confirmatory Stages.** Tukey and Mallows [50] clearly lay out the modern paradigm of separate diagnostic data analysed for the confirmatory (testing) stages of a data analysis. We all know we should not be double dipping the data. The ideal situation is to do the *detective work* and exploratory data analysis (EDA) on one set of data; once this data has told us which variables to use, which transformations to make and which dimension reductions to implement, the critical and confirmatory analysis is done on a different set. Setting up the confirmatory analyses, with fixed algorithms and computations untouched by human hand (Donoho, personal communication) would be ideal and guarantee the integrity of the conclusions. However, applied statisticians, as opposed to mathematicians, do not live in an ideal world.

5.3. **Honest appraisal of current progress.** In the early 1980's, Diaconis [19] had carefully laid out suggestions for best practices for organizing exploratory analyses in order to avoid some of the pitfalls exposed above. It is certainly the case that transparency should be a priority. Authors should say clearly: what are the assumptions about the data that are used to find the null distribution?
Was optimization over a certain set of possibilities used? Can an uncertainty measure be attached to an estimate?
Today, we know the biases created by pressure to publish incite p-hacking [65]. We also know, this can be avoided if authors are encouraged to publish all the variables to compute them and disclose all the relevant information. The original experimental design should be registered, the exploratory analysis performed on one set of data, a clear algorithm decided upon before the collection of a separate confirmatory set. The confirmatory analysis should be published or posted whether it is positive or not.

Many authors have published papers *talking the talk*: offering ideas on the careful prescriptions to follow, there are many papers published doing meta-analyses of all the publications and showing how much new approaches are needed. However, there are many more papers giving lists of prescriptions to heal the problem than papers available showing scientists *actually walking the walk*. This is because the incentive structure has not been changed, publishing one's data and code is costly in terms of recognition, time and money. From experience, I know that a complete data workflow paper with a confirmatory data set takes almost a year more to publish [20, 16]. This is not a problem where mathematics can help but mathematicians have set an example and a publication record where transparency is key and theorems are proved, mathematicians *show their work* and so should statisticians.

Currently, we have no excuse. Open access is here, whole toolkits of open source statistical packages such as Bioconductor [33], Galaxy [28], IPython notebooks [52] and other such systems are available and publishing code, data and the complete workflows should be standard.

Multiplicity but not duplicity. Everything we presented above addresses errors that are made in good faith. In fact if the data are dredged secretly, dropping points which do not support the narrative the scientists wants to publish or biasing the sampling, none of the approaches presented above can work. Baggerly and Coombs [2] showed by doing careful bioinformatic and statistical forensics that certain studies in the world of cancer genetics were flawed with intent to dupe. No statistical procedure can counter data manipulations done in bad faith, all we can hope for is to achieve more transparency through careful accounting for, and recording of, all the choices made during an analysis.

Those are the cases where true scientific replication by an independent group and funded by the science foundations and institutes should find their place in mainstream publication.

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[6]The code to reproduce the analyses and figures in this article is available at: http://statweb.stanford.edu/~susan/papers/RR/.

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81. Peter H Westfall, A Krishen, and Stanley S Young, *Using prior information to allocate significance levels for multiple endpoints*, Stat. Med. **17** (1998), no. 18, 2107–2119.
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January 8, 2016 (Seattle, WA)

Carina Curto, Pennsylvania State University
What can topology tell us about the neural code?

Lionel Levine, Cornell University and *Yuval Peres, Microsoft Research
and University of California, Berkeley
Laplacian growth, sandpiles and scaling limits.

Timothy Gowers, Cambridge University
Probabilistic combinatorics and the recent work of Peter Keevash

Amie Wilkinson, University of Chicago
What are Lyapunov exponents, and why are they interesting?

January 12, 2015 (San Antonio, TX)

Jared S. Weinstein, Boston University
Exploring the Galois group of the rational numbers: Recent breakthroughs.

Andrea R. Nahmod, University of Massachusetts, Amherst
*The nonlinear Schrödinger equation on tori: Integrating harmonic analysis,
geometry, and probability.*

Mina Aganagic, University of California, Berkeley
String theory and math: Why this marriage may last.

Alex Wright, Stanford University
From rational billiards to dynamics on moduli spaces.

January 17, 2014 (Baltimore, MD)

Daniel Rothman, Massachusetts Institute of Technology
Earth's Carbon Cycle: A Mathematical Perspective

Karen Vogtmann, Cornell University
The geometry of Outer space

Yakov Eliashberg, Stanford University
Recent advances in symplectic flexibility

Andrew Granville, Université de Montréal
Infinitely many pairs of primes differ by no more than 70 million (and the bound's getting smaller every day)

January 11, 2013 (San Diego, CA)

Wei Ho, Columbia University
How many rational points does a random curve have?

Sam Payne, Yale University
Topology of nonarchimedean analytic spaces

Mladen Bestvina, University of Utah
Geometric group theory and 3-manifolds hand in hand: the fulfillment of Thurston's vision for three-manifolds

Lauren Williams, University of California, Berkeley
Cluster algebras

January 6, 2012 (Boston, MA)

Jeffrey Brock, Brown University
Assembling surfaces from random pants: the surface-subgroup and Ehrenpreis conjectures

Daniel Freed, University of Texas at Austin
The cobordism hypothesis: quantum field theory + homotopy invariance = higher algebra

Gigliola Staffilani, Massachusetts Institute of Technology
Dispersive equations and their role beyond PDE

Umesh Vazirani, University of California, Berkeley
How does quantum mechanics scale?

January 6, 2011 (New Orleans, LA)

Luca Trevisan, Stanford University
Khot's unique games conjecture: its consequences and the evidence for and against it

Thomas Scanlon, University of California, Berkeley
Counting special points: logic, Diophantine geometry and transcendence theory

Ulrike Tillmann, Oxford University
Spaces of graphs and surfaces

David Nadler, Northwestern University
The geometric nature of the Fundamental Lemma

January 15, 2010 (San Francisco, CA)

Ben Green, University of Cambridge
Approximate groups and their applications: work of Bourgain, Gamburd, Helfgott and Sarnak

David Wagner, University of Waterloo
Multivariate stable polynomials: theory and applications

Laura DeMarco, University of Illinois at Chicago
The conformal geometry of billiards

Michael Hopkins, Harvard University
On the Kervaire Invariant Problem

January 7, 2009 (Washington, DC)

Matthew James Emerton, Northwestern University
Topology, representation theory and arithmetic: Three-manifolds and the Langlands program

Olga Holtz, University of California, Berkeley
Compressive sensing: A paradigm shift in signal processing

Michael Hutchings, University of California, Berkeley
From Seiberg-Witten theory to closed orbits of vector fields: Taubes's proof of the Weinstein conjecture

Frank Sottile, Texas A & M University
Frontiers of reality in Schubert calculus

January 8, 2008 (San Diego, California)

Günther Uhlmann, University of Washington
Invisibility

Antonella Grassi, University of Pennsylvania
Birational Geometry: Old and New

Gregory F. Lawler, University of Chicago
Conformal Invariance and 2-d Statistical Physics

Terence C. Tao, University of California, Los Angeles
Why are Solitons Stable?

January 7, 2007 (New Orleans, Louisiana)

Robert Ghrist, University of Illinois, Urbana-Champaign
Barcodes: The persistent topology of data

Akshay Venkatesh, Courant Institute, New York University
Flows on the space of lattices: work of Einsiedler, Katok and Lindenstrauss

Izabella Laba, University of British Columbia
From harmonic analysis to arithmetic combinatorics

Barry Mazur, Harvard University
The structure of error terms in number theory and an introduction to the Sato-Tate Conjecture

January 14, 2006 (San Antonio, Texas)

Lauren Ancel Myers, University of Texas at Austin
Contact network epidemiology: Bond percolation applied to infectious disease prediction and control

Kannan Soundararajan, University of Michigan, Ann Arbor
Small gaps between prime numbers

Madhu Sudan, MIT
Probabilistically checkable proofs

Martin Golubitsky, University of Houston
Symmetry in neuroscience

January 7, 2005 (Atlanta, Georgia)

Bryna Kra, Northwestern University
The Green-Tao Theorem on primes in arithmetic progression:
A dynamical point of view

Robert McEliece, California Institute of Technology
Achieving the Shannon Limit: A progress report

Dusa McDuff, SUNY at Stony Brook
Floer theory and low dimensional topology

Jerrold Marsden, Shane Ross, California Institute of Technology
New methods in celestial mechanics and mission design

László Lovász, Microsoft Corporation
Graph minors and the proof of Wagner's Conjecture

January 9, 2004 (Phoenix, Arizona)

Margaret H. Wright, Courant Institute of Mathematical Sciences,
New York University
The interior-point revolution in optimization: History, recent
developments and lasting consequences

Thomas C. Hales, University of Pittsburgh
What is motivic integration?

Andrew Granville, Université de Montréal
It is easy to determine whether or not a given integer is prime

John W. Morgan, Columbia University
Perelman's recent work on the classification of 3-manifolds

January 17, 2003 (Baltimore, Maryland)

Michael J. Hopkins, MIT
Homotopy theory of schemes

Ingrid Daubechies, Princeton University
Sublinear algorithms for sparse approximations with excellent odds

Edward Frenkel, University of California, Berkeley
Recent advances in the Langlands Program

Daniel Tataru, University of California, Berkeley
The wave maps equation

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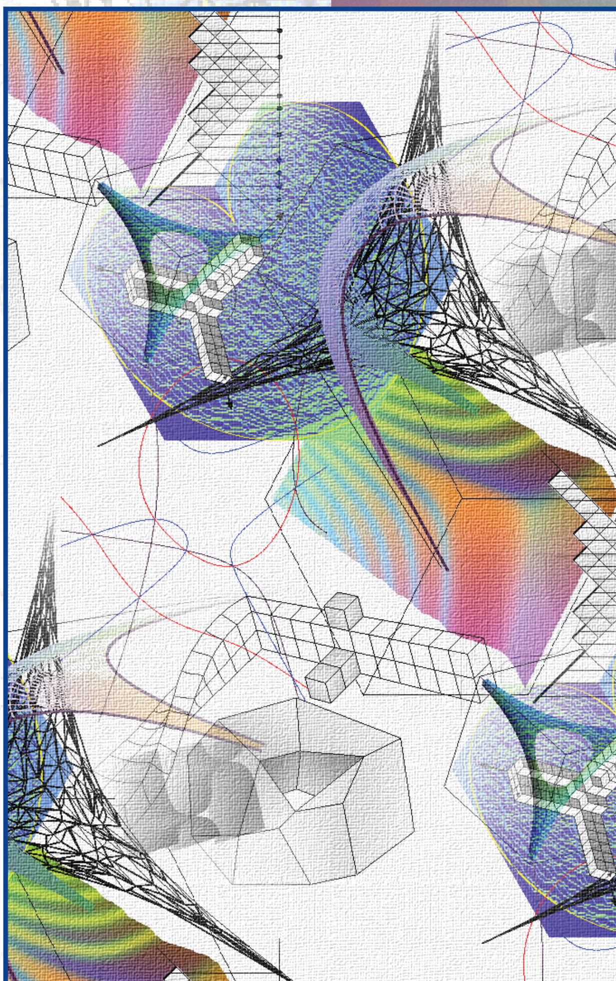

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