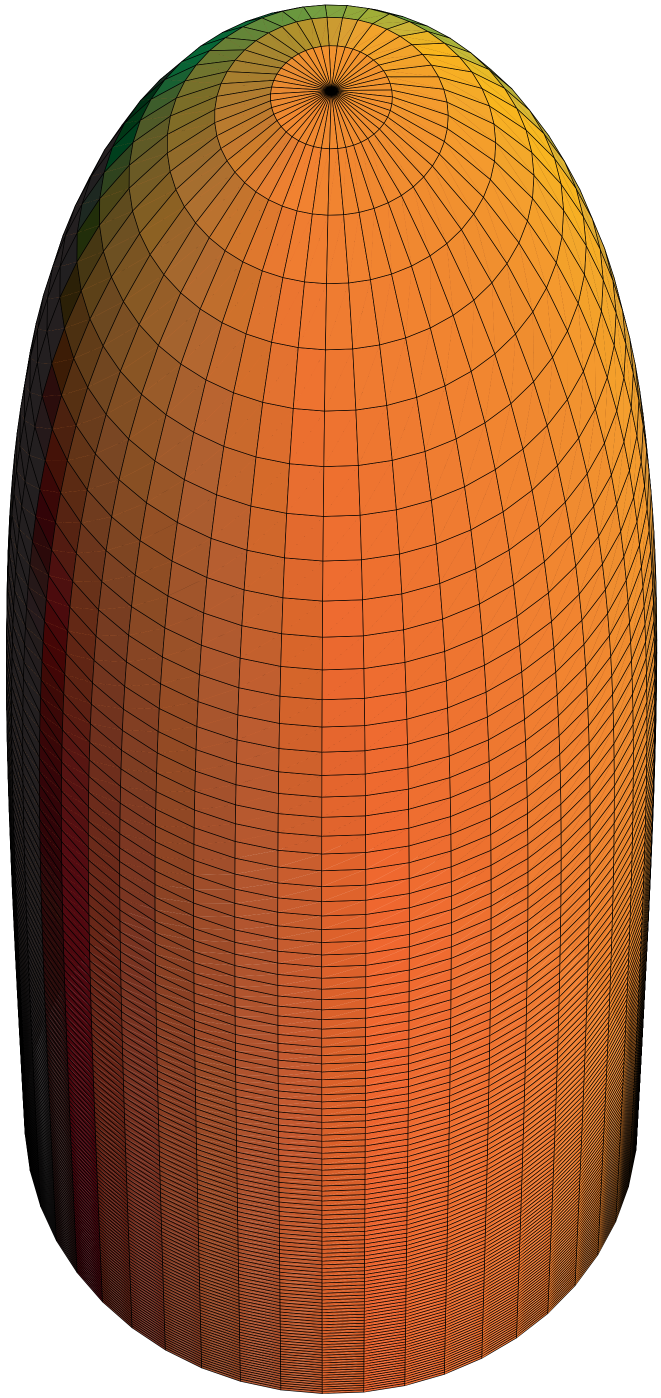

The "cigar." *This is a solution of the Ricci flow equation that remains stationary for all time, and therefore does not form a narrow tendril that can be broken off by surgery. One of the first key steps of Perelman's proof was to show that the cigar does not arise as a limit of Ricci flow on three-dimensional manifolds. (Graphic created by Michael Trott using Mathematica.)*

First of Seven Millennium Problems Nears Completion

Barry Cipra

IN THE MILLENNIAL YEAR OF 2000, the Clay Mathematics Institute focused the world's attention on seven mathematical problems of exceptional historical and practical interest, by offering a million-dollar prize for the solver of any of them. (See "Think and Grow Rich," *What's Happening in the Mathematical Sciences*, Volume 5). The seven problems were designated Millennium Prize Problems. Within three years, the first serious contender for one of the million-dollar prizes emerged. It now appears likely that the list of Millennium Problems will shrink from seven to six before the end of the millennium's first decade.

In November 2002 and March 2003, Grigory Perelman of the Steklov Institute in Moscow posted a pair of papers that outlined the key steps in settling a century-old topological problem known as the Poincaré conjecture. Because the two papers bring together ideas from disparate fields, and leave many details for the reader to fill in, experts have found them to be tough going. However, in more than three years of scrutiny, none of them have found any gaps that seriously jeopardized Perelman's claim to proving the theorem. Several experts seem to be cautiously edging towards pronouncing the proof complete.

Perhaps just as importantly, Perelman has introduced a raft of new techniques in differential geometry, which experts expect will revolutionize the field. These techniques may prove sufficient to prove an even broader result in topology, known as Thurston's geometrization conjecture. For the subject of three-dimensional topology, this would be as fundamental a result as the discovery of the periodic table.

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A Topological History Tour

What are these conjectures, and why are they so important? The answer calls for a little background.

In the nineteenth century, mathematicians came to grips with the mathematical nature of surfaces. They discovered a simple scheme for classifying surfaces using a single number, the so-called genus. Two surfaces are topologically identical—meaning it's possible to deform one into the other—if and only if they have the same genus.

The simplest example of a surface is, of course, the flat, Euclidean plane. But topologists prefer to work with surfaces that are closed and bounded—the term of art is "compact"—so the

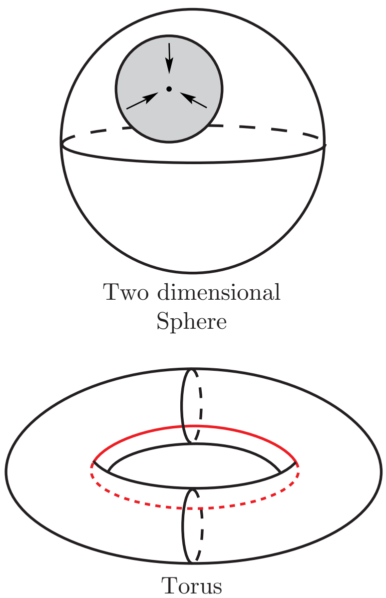


Figure 1. *A two-dimensional sphere is simply connected because any loop on the surface of the sphere can be tightened down to a point. On the other hand, a torus is not simply connected. The red loop shown here is caught around a neck of the torus and cannot be pulled any tighter.*

theory focuses instead on the sphere, which can be thought of as the plane with an extra "point at infinity" (see Figure "Stereographic Projection" in "Combinatorics Solve a Venn-erable Problem," p. 51). The sphere has genus zero. Roughly speaking, this means it has no "hole." Somewhat more precisely, it means that any closed loop drawn on the sphere can continuously shrink to a single point (see Figure 1 (top)).

The genus is always a non-negative integer, and there is a compact surface of each genus. A genus-1 surface has a single hole, and looks more or less like a doughnut. Such a surface is called torus. Because it has genus 1, not every closed loop drawn on the torus can shrink to a single point—a loop that goes around the hole, for example, cannot (see Figure 1 (bottom)). Surfaces of higher genus have correspondingly more holes, but the theory is still simple: If two compact surfaces have the same number of holes, then they are, topologically speaking, the same.

The mathematical nature of compact three-dimensional spaces, or 3-manifolds as they're called, is much more complicated. There is no simple number akin to the genus that distinguishes one 3-manifold from another. Researchers have instead developed a panoply of techniques for studying 3-manifolds.

The modern theory of 3-manifolds was initiated around 1900 by the French mathematician Henri Poincaré. Poincaré developed an algebraic theory in which closed curves in a 3-manifold are associated with elements in a mathematical group, called the manifold's *fundamental group*. It was a crucial insight for the emergence of topology as a viable branch of mathematics. Manifolds are floppy, hard to draw and hard to visualize. Algebra, on the other hand, is concise, precise, and lends itself to symbolic manipulation. If a manifold could be uniquely described by a group, the theory of manifolds would become vastly simpler.

Alas, such a neat correspondence was not to be. It's true that if two manifolds have different fundamental groups, the manifolds are topologically different. However, the converse is not true: Two manifolds can have the same fundamental group yet still be different.

In a paper published in 1904, Poincaré conjectured an important exception to this lack of a converse: If the fundamental group of a 3-manifold is trivial—which happens when every closed curve can shrink to a point—then that manifold is identical to the simplest possible 3-manifold, known as the 3-sphere.

The 3-sphere is a three-dimensional analogue of the ordinary two-dimensional sphere (sometimes called the 2-sphere). Just as the 2-sphere can be thought of as the Euclidean plane with an extra point at infinity, the 3-sphere can be thought of as Euclidean space with an extra point at infinity. It can also be viewed as the solution set in 4-dimensional space of the equation $x^2+y^2+z^2+w^2 = 1$, just as the 2-sphere is the solution set in 3-space of the equation $x^2 + y^2 + z^2 = 1$ (and the "1-sphere," or circle, is the solution set in the plane of the equation $x^2+y^2 = 1$—in general, the n-sphere is the solution set of the equation $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$).

Actually Poincaré didn't phrase the conjecture as a conjecture. He posed it as a question: Is it possible for a 3-manifold to have trivial fundamental group without being identical with the 3-sphere? (The technical term for "identical" is "homeomorphic," which derives from the Greek for "same shape.") The difference between a conjecture and an open question is important. To pose a statement as a conjecture, one should have strong but not necessarily convincing evidence that it is true. Poincaré's caution may have been been prompted by his own bitter experience. Four years earlier he had stated, as a theorem, a stronger claim that turned out to be false; his 1904 paper retracted that claim and gave a counterexample. In his earlier work, Poincaré was studying a different and somewhat less powerful algebraic structure associated with manifolds, known as a homology group. He had falsely claimed that an n-manifold—for any dimension n—is homeomorphic to the n-sphere if it has the same homology groups as the n-sphere. His 1904 counterexample was a 3-manifold with the same homology groups as the 3-sphere but with a nontrivial fundamental group.

Poincaré was the first of many people to be fooled by the elusive 3-sphere.

Poincaré was the first of many people to be fooled by the elusive 3-sphere. There were numerous attempts in the twentieth century to prove the Poincaré conjecture, and several claims to have done so. Like Fermat's Last Theorem (see "Fermat's Theorem—At Last!," *What's Happening in the Mathematical Sciences*, Volume 3), it came to occupy mathematicians' short list of notorious problems—seemingly simple problems that nevertheless mocked the attempts of even experienced mathematicians to solve them.

As mathematicians grew more interested in higher-dimensional manifolds (which cannot be visualized in our three-dimensional universe, but which are nevertheless every bit as "real" to a topologist), they naturally wondered whether the Poincaré conjecture could be extended to these manifolds as well. If an n-dimensional, compact manifold had a trivial fundamental group, was it necessarily an n-sphere?

At first blush, it might seem that the n-dimensional version of the Poincaré Conjecture must be much harder than the 3-dimensional version. After all, we can't even see what an n-dimensional space looks like. But the first major breakthrough on the Poincaré Conjecture came in 1960, when Stephen Smale of the Institute for Advanced Study and John Stallings at Oxford University independently proved that it was true for manifolds of 5 or more dimensions. Two decades later, in 1982, Michael Freedman, now at Microsoft Research in Bellevue, Washington, proved the conjecture for $n = 4$. As a result of the deep theorems of Smale, Stallings, and Freedman, mathematicians now knew how to identify the n-sphere in every case *except* the one Poincaré had originally asked about: the case $n = 3$.

Unfortunately, there seemed to be no chance that the proofs that worked in higher dimensions could somehow be adapted to three dimensions. The extra degrees of freedom available in higher dimensions sometimes permit techniques that don't work in lower dimensions. For example, in 4 and higher dimensions, every curve can be unknotted. But as we all know from experience, knots do exist in 3-space. Thus, for example, any proof of the Poincaré Conjecture that involved untying knots would not be valid in 3 dimensions. An obstacle very much like this (only involving two-dimensional "knots") is a principal reason why Smale's and Stallings' proofs could not work in dimensions lower than 5.

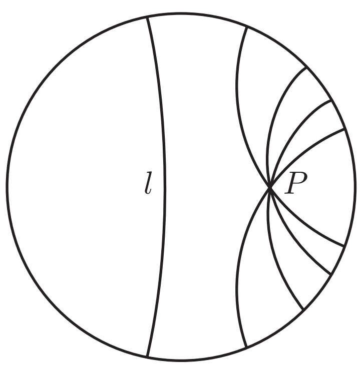
Life in Three Dimensions

Three-dimensional manifolds have a very different "flavor" from high-dimensional ones. In the 1970s, William Thurston, then at Princeton University, realized that geometry plays a decisive role in low-dimensional topology, and in fact his geometric approach now dominates the subject. Thurston proposed what's called the geometrization conjecture. Very roughly speaking, geometry is topology with a notion of distance and angle; if topology is "rubber sheet" geometry, then geometry is "crystallized" topology. Thurston conjectured that every 3-manifold can be cut into pieces so that each piece will "freeze-dry" into a geometric structure associated with one of eight possible three-dimensional geometries. (See Figure 4, p. 11.) These geometries are well understood. In particular, only one of them has a trivial fundamental group: the 3-sphere.

Parts of Thurston's conjecture are incontrovertible. One is the existence of precisely eight geometries. A geometric structure, in Thurston's theory, is a compact three-dimensional space with a special way of measuring angles and distance known as a homogeneous Riemannian metric (named for the nineteenth-century mathematician Bernhard Riemann, who initiated the abstract study of manifolds). "Homogeneous" means the metric is the same at every point, much as homogenized milk has the same consistency throughout. The associated "geometry" refers to another three-dimensional space, usually non-compact, with the same metric, but with trivial fundamental group. (In the general theory of manifolds, every manifold, geometric or not, has a "universal covering manifold" which has trivial fundamental group. Because the universal covering manifold is not necessarily compact, it is not necessarily a sphere.) The geometric structure can be identified as a "quotient" of its associated geometry by a discrete group of motions, which is algebraically the same as the structure's fundamental group. Thus geometry, in Thurston's program, provides the link between topology and algebra.

The notion of geometrization is easier to understand in two dimensions, where something similar occurs. For surfaces, there are three different geometries: the sphere, the familiar Euclidean plane, and the less familiar, non-Euclidean, "hyperbolic" plane. The sphere is its own covering space; its group of motions consists of rotations. The Euclidean plane is the geometry for the torus: Since the torus can be viewed as a square with opposite edges identified, copies of it naturally tile the plane. The group of motions for the plane consists of translations and rotations.

Everything else—each surface of genus 2 and higher—is associated with the hyperbolic plane. The hyperbolic plane has a simple model: the interior of a circle. But instead of "lines" being lines in the usual, Euclidean sense, a hyperbolic line is a circular arc that meets the perimeter of the circle at right angles. This is what makes the hyperbolic plane non-Euclidean:



Poincaré model

Figure 2. *In the Poincaré disk model of hyperbolic geometry, the hyperbolic plane is represented by the inside of a disk, and lines are represented by circular arcs that are perpendicular to the boundary of the disk. Given any line l and any point P not on that line, there are many parallel lines to l through P, such as the four shown in this figure. This property distinguishes hyperbolic geometry from Euclidean geometry, where there is only one parallel to a given line through any point not on that line.*

Given a "line"and a point not on the "line," there is not just one "line" through the point that is "parallel" to the given "line"—there are, in fact, infinitely many "lines" through the point that don't intersect the given "line" (see Figure 2, opposite page).

The group of motions of the hyperbolic plane also consists of translations and rotations, but it permits tilings of the hyperbolic plane that are impossible in the Euclidean plane (see Figure 3, pages 8 and 9). This is what allows the hyperbolic plane to act as the geometry for higher-genus surfaces. For example, the hyperbolic plane can be tiled by right-angled octagons. (That is not possible in the Euclidean plane because right-angled octagons don't exist in Euclidean geometry.) Every motion of the hyperbolic plane that keeps the tiling intact corresponds to an element of the fundamental group of a two-holed torus. Thus the two-holed torus has a hyperbolic geometry.

A crucial aspect of these geometries is that they come with a way of measuring distance (called a Riemannian metric), and they endow their respective manifolds with a quality called curvature. The sphere has positive curvature, the Euclidean plane has zero curvature, and the hyperbolic plane has negative curvature. Because the metric is homogeneous, the curvature is the same at every point. For the sphere and hyperbolic space, it is usually normalized to 1 and -1, respectively.

The curvature of the sphere and the Euclidean plane are easy to understand. That of the hyperbolic plane is less clear. Curvature fundamentally has to do with what happens to nearby curves in a surface if they start out parallel and continue as straight as possible while remaining in that surface. In Euclidean geometry, they remain equidistant. In a surface of positive curvature, like the sphere, they move towards each other. (For example, think of two curves that start from the equator and head due south, moving as straight as possible *without leaving the surface of the sphere.* The two curves will always intersect at the south pole.) In hyperbolic geometry, on the other hand, curves that start out parallel tend to get farther and farther apart.

In sum, every surface is associated with one of three geometries. The 2-sphere (genus 0) is associated with itself. The torus (genus 1) has the flat geometry of the Euclidean plane. And surfaces of higher genus have a hyperbolic geometry. In effect, Thurston's geometrization conjecture says that much the same is true for 3-manifolds. Only there are eight geometries, not three. And the manifold may need to be cut into pieces first.

Three of the eight geometries are 3-D analogs of their 2-D counterparts. Spherical geometry is just the 3-sphere. Euclidean geometry is what you think it is. Hyperbolic geometry is a 3-D version of the hyperbolic plane, with the interior of the sphere acting as a model, much as the interior of a circle is a model for the hyperbolic plane. Two others are "product spaces" of the line with the sphere and the hyperbolic plane, respectively. These geometries are flat in one direction and curved in the other two. The final three geometries, called Solv, Nil, and $SL(2, R)$, are specialized; they occur rather rarely, but need to be taken in account for the sake of completeness. (This is perhaps a bit unfair. It's a little like dismissing 2 as an unimportant prime number just because even primes are so rare.)

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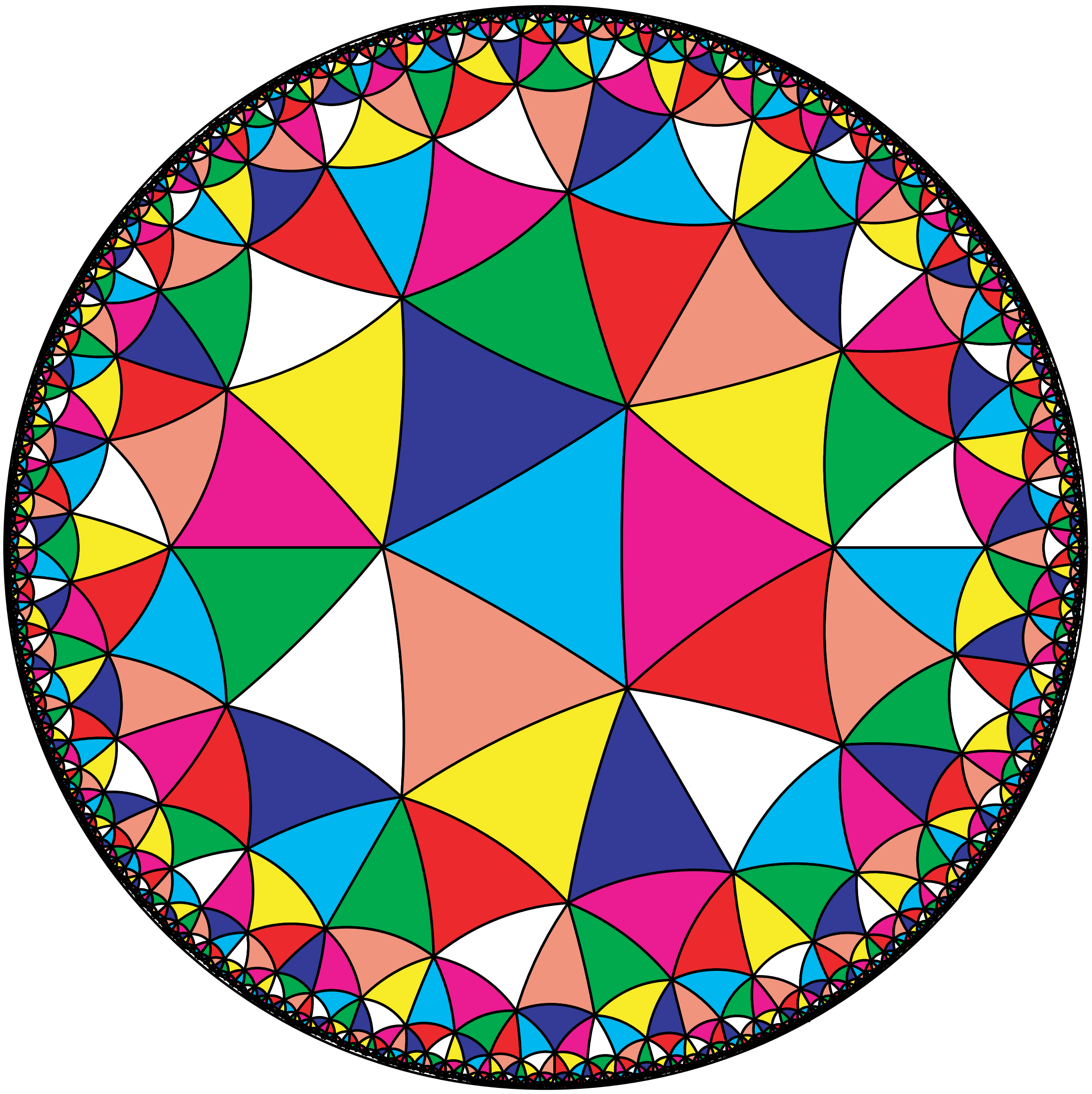

Figure 3a. *For caption, see Figure 3b on the next page. (Figure courtesy of Douglas Dunham.)*

Geometrization is a powerful concept because the structure imposed by a homogeneous Riemannian metric makes many topological properties much more accessible. In particular, the only geometry that permits fundamental groups of finite size is the spherical geometry— and the only one with trivial fundamental group is the 3-sphere itself. In other words, the Poincaré conjecture is an "easy" consequence of Thurston's geometrization conjecture.

Thurston and others have proved chunks of the geometrization conjecture. Thurston, for example, showed that a large class of 3-manifolds known as Haken manifolds (named after Wolfgang Haken, who is best known for his 1976 proof, with Ken Appel, of the famous four-color theorem) all have the geometry of hyperbolic space. (In a certain sense, "most" 3-manifolds are hyperbolic.)

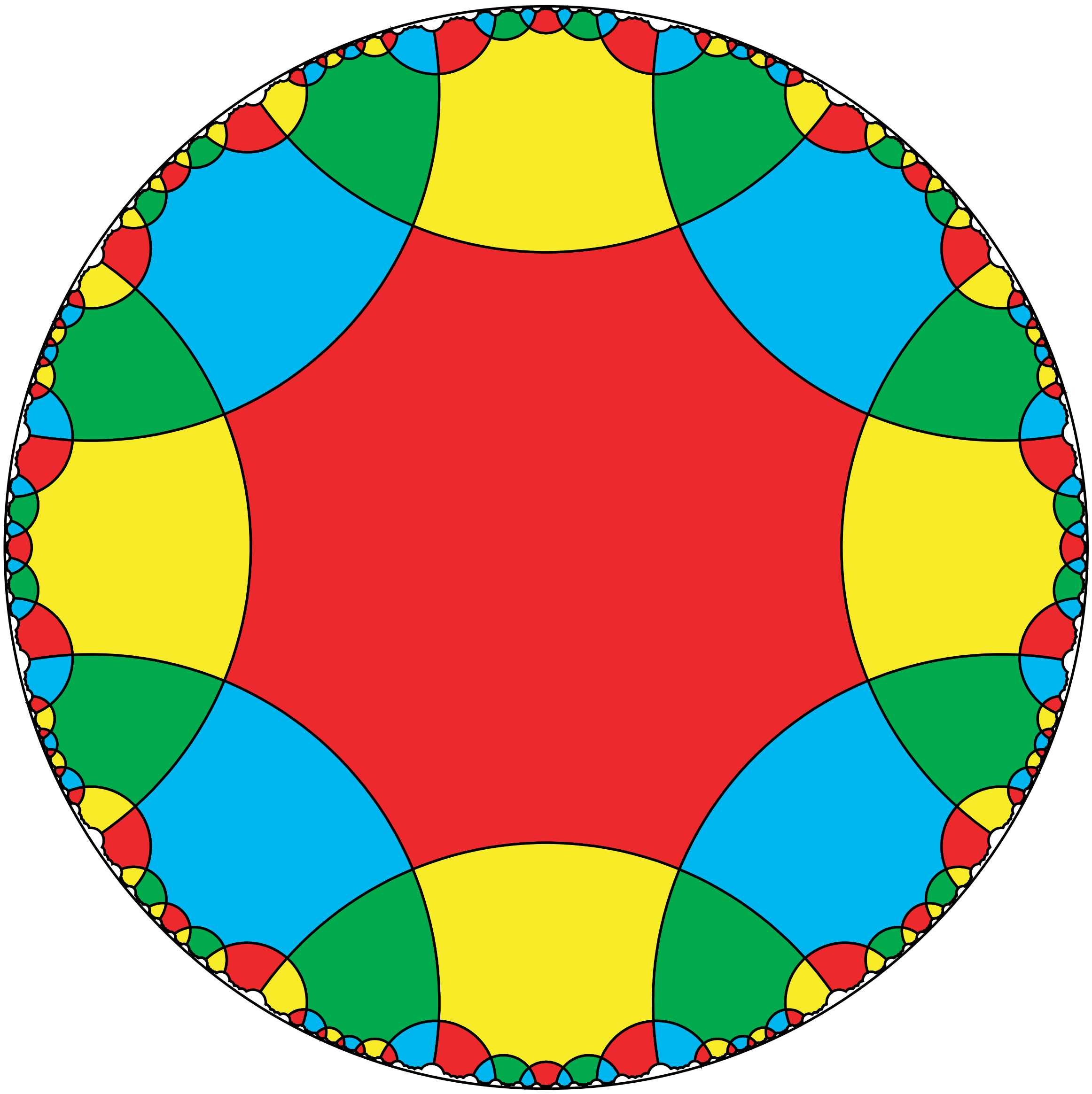

Figure 3b. *The hyperbolic plane has many more tessellations by regular polygons than the Euclidean does. These figures show two of them: a tessellation by equilateral triangles in which seven triangles meet at each vertex (a), and a tessellation by octagons in which four octagons meet at each vertex (b). These figures are not possible in Euclidean geometry because the sum of the angles meeting at each vertex of the tessellation would be greater than* 2π *radians. (Figure courtesy of Douglas Dunham.)*

Going with the (Ricci) Flow

In the early 1980s, Richard Hamilton, then at the University of California at San Diego, proposed a way of proving Thurston's geometrization conjecture using a concept that came to be called Ricci flow. Every manifold (in any dimension) comes equipped with some sort of Riemannian metric. The problem is, the metric may not be homogeneous. Hamilton's idea was to let the manifold "find its own way" to a homogeneous metric, by allowing the metric to evolve in a way very reminiscent of the way that heat flows. If heat flows unimpeded through a room

The "Ricci oven" does all the work!

with no heat sources or sinks, the temperature will eventually equilibrate. If the same thing could be done with a manifold, the curvature would eventually equilibrate—and the manifold would become homogeneous.

Heat flow is described by a partial differential equation that equates the rate of change of temperature in time to its variation (technically, its second derivative) in space. Hamilton was led to a similar equation for Riemannian metrics: Let the rate of change of the metric in time be proportional to a quantity called its Ricci curvature, named for the nineteenth-century Italian mathematician Gregorio Ricci-Curbastro. Ricci curvature is closely related to the notion of space-bending curvature in Einstein's general theory of relativity. Hamilton's Ricci flow equation has a very simple form: If $g = g(t)$ denotes the Riemannian metric at time t, then $\partial g/\partial t = -2R(g)$, where R is the Ricci curvature.

Hamilton's idea, in a nutshell, is that if a 3-manifold is in fact geometric, then no matter what Riemannian metric is placed on it to begin with, the Ricci flow equation will smooth it out so that the curvature is the same everywhere. In other words, starting from an amorphous, topological shape, the geometry of a 3-manifold will reveal itself through the limit of $g(t)$ as t goes to infinity.

John Morgan, a topologist at Columbia University, uses a cooking metaphor to describe the Ricci flow (and its connection with the heat equation): Imagine your manifold as a dollop of cookie dough, and the Ricci flow equation as an oven. The blob that goes into the oven has no particular shape, but what comes out is a nicely rounded—and crisp—cookie. The "Ricci oven" does all the work!

What if the manifold is a compound of two different geometries? Metaphorically speaking, what if it's an amalgam of cookie dough, pancake batter, and muffin mix? Ideally, the Ricci oven will cause the different pieces to separate, so that after a while the constituents are easily identifiable, with only a few tendrils keeping them connected—and eventually even the tendrils will snap or evaporate (see Figure 5, page 12).

Metaphor, however, is not proof. Hamilton's idea ran into theoretical obstacles. The main difficulties had to do with the nature of the tendrils—more technically known as singularities—that develop as pieces of a manifold try to separate. There seemed to be nothing preventing a thick tangle of singularities from developing, perhaps to the point that the metaphoric manifold would become all tendril and no cookie. Nor was it clear what kind of singularities were possible. In particular, Hamilton and others working on Ricci flow were not able to rule out a type of singularity they called the "cigar" (see Figure on page 2, "The Cigar"). The cigar, also known as Witten's black hole, is a rotationally symmetric solution of the Ricci flow equation for the (non-compact) Euclidean plane. In essence, it is a singularity that is perfectly happy "cooking" in the Ricci oven for all time. This would defeat the plan of having all the singularities break or evaporate.

Singularities don't necessarily show up at all. Hamilton showed the Ricci oven works perfectly on manifolds of positive curvature: Even if some parts are curved more than others,

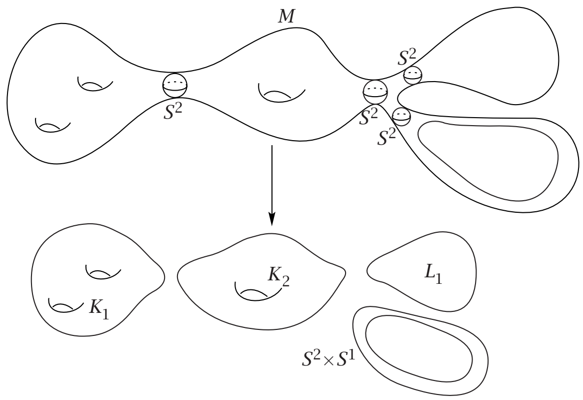

Figure 4. *Thurston's geometrization conjecture posited that any three-dimensional manifold can be decomposed into pieces that each have one of eight geometric structures. This figure illustrates pieces with four different structures: a two-holed torus would have a hyperbolic structure, a one-holed torus would have a Euclidean structure, a sphere would have a spherical structure, and a Seifert fibered manifold such as $S^2 \times S^1$ has a product structure. The conjecture allows some of the cuts to be made along tori (not shown here) rather than along spheres. For purposes of illustration the pieces are drawn as two-dimensional surfaces, but in reality they would be three-dimensional. (Reprinted from "Geometrization of 3-Manifolds via the Ricci Flow," by Michael Anderson, AMS Notices, February 2004, Figure 1, page 185.)*

the Ricci flow equation smooths things out so that the manifold asymptotically develops constant curvature. This showed that all such manifolds are geometric. Hamilton also proved that singularities take time to develop: If the metric is smoothly curved to begin, then it stays smoothly curved at least for a while. But it was clear that singularities do develop in the presence of negative curvature—and once they do, there was no telling what would happen.

Perelman changed all that. In a pair of "e-prints" posted online, the Russian mathematician laid out a program for dealing with the singularities that arise in Ricci flow. In the first paper, he analyzed the nature of the singularities and showed that the cigar (for example) does not occur. According to Morgan, experts realized very quickly that this part of Perelman's proof was for real, and this gave them motivation to tackle the much more difficult parts that followed. One of the key ingredients in the analysis is a notion of entropy for metrics. In thermodynamics entropy is a measure of disorder, which tends to increase over time. Perelman's metric entropy has a similar propensity to increase, which is what keeps the singularities under control.

Perelman's second paper shows how to deal with singularities as they arise. The basic idea is to cut out pieces of the manifold that are neighborhoods of developing singularities, glue in smooth pieces that are constructed by hand, and then let the Ricci flow continue to act on the modified manifold. One concern is that the cutting (topologists call it surgery, because it's actually not enough to cut—you also have to sew up the wound!)

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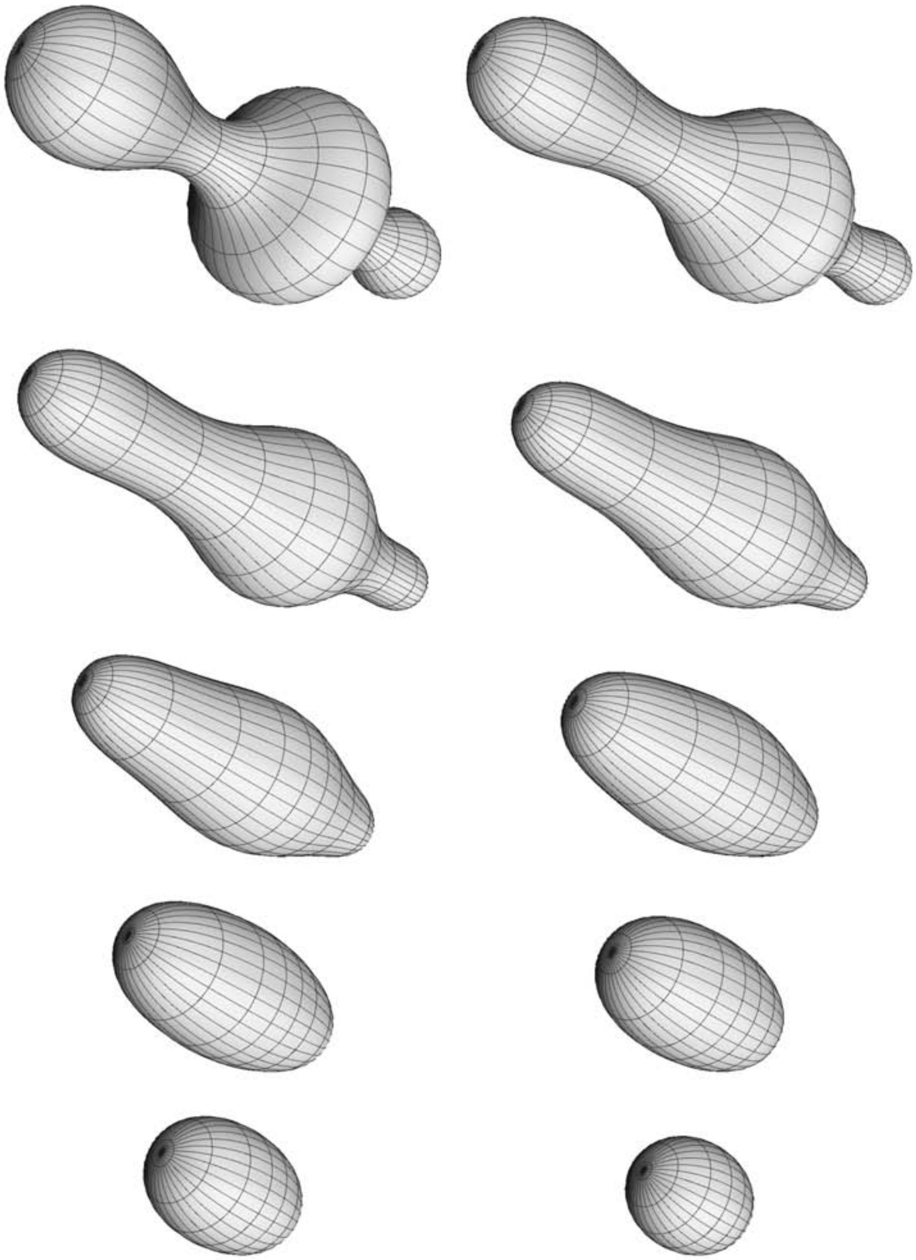

Figure 5. *Ricci flow on 2-dimensional manifolds tends to make the curvature more uniform, so they approach a sphere, a torus, or a multi-holed torus. This figure shows how an initially bumpy surface smoothes out to form a sphere. Ricci flow in 3 dimensions has the same tendency to even out the curvature, but it is greatly complicated by the fact that singularities such as narrow necks or horns can form and pinch off. (Courtesy of J. Hyam Rubinstein of the University of Melbourne and Robert Sinclair of the Okinawa Institute of Science and Technology.)*

could cause even more singularities to arise, so that one would wind up with an infinite number of pieces in a finite amount of time. Perelman overcame this with another argument reminiscent of the entropy idea. He showed that the surgery could be done in such a way as to reduce the volume of the manifold by a certain amount, $\epsilon(T)$. At the same time, the Ricci oven causes the volume to grow, just as a loaf of bread grows in a real oven. But if there were infinitely many surgeries in a finite time, they would subtract an infinite volume from the manifold—and the growth of volume due to the Ricci flow would not be enough to compensate. Therefore, only finitely many surgeries occur in any finite time interval. Thus the flow-with-surgery process can continue for all time, and that is enough time for the manifold to separate into pieces with distinct homogeneous geometries.

Perelman's work sets the stage for a proof of Thurston's geometrization conjecture. At the end of the second paper, he outlined how the argument goes and promised to give details in a third paper. That paper is yet to appear. However, Perelman posted a very short paper in July 2003 that gives a separate, simpler proof of the Poincaré conjecture, based on the results of the first two papers. At roughly the same time, Tobias Colding at the Courant Institute of Mathematical Sciences and William Minicozzi II at Johns Hopkins University gave another argument, also based on Perelman's two papers, to the same effect.

The simplicity of both proofs gave researchers confidence that the Poincaré conjecture at least was within their grasp—provided there were no lacunae in Perelman's main papers. That may seem like a simple matter of refereeing. But Perelman's work has kept experts busy for more than three years. The arguments are extremely technical, and there are a multitude of new ideas in the proof. Three separate teams of mathematicians have now produced book-length manuscripts explicating Perelman's proof. One team, Bruce Kleiner and John Lott of the University of Michigan, posted their work on the Internet as they progressed. The second team, Huai-Dong Cao of Lehigh University and Xi-Ping Zhu of Zhongshan University in China, published their paper in the June 2006 issue of *Asian Journal of Mathematics*. The third team, Morgan and Gang Tian of the Massachusetts Institute of Technology, plans to publish their manuscript as a book. Morgan says that he is convinced, but that does not mean that the story is over. "The experts are very optimistic but cautious. If this were an ordinary problem, we would have been satisfied two years ago. We have only continued [to question it] because it's the Poincaré Conjecture and it has such a long history of mistakes, even by very good mathematicians."

"If this were an ordinary problem, we would have been satisfied two years ago."

To be eligible for the Millennium Prize, a solution of the Poincaré Conjecture must be published in a refereed journal "or other such form as the Science Advisory Board [of the Clay Mathematics Institute] shall determine qualifies" and to survive two years of further scrutiny after that. Perelman himself has been conspicuously silent since releasing his preprints, and has never submitted them to a journal. However, James Carlson, the president of the Clay Mathematics Institute, has confirmed that the three exegeses of Perelman's work satisfy the publication requirement, and the two-year clock is now "ticking." Thus it seems likely that mathematicians' first great breakthrough of the third millennium will be ready to enter the history books sometime in 2008 or 2009.