Hall of Mirrors. *Spectacular images result from plotting the action of a Kleinian group in 3-dimensional space. To a topologist, this image represents the outside, or convex hull, of a 3-dimensional "hall of mirrors." Each individual room in the hall of mirrors corresponds to a 3-dimensional manifold. In a case like this one, where the walls of the hall of mirrors form visible spheres, the manifold is said to be "geometrically finite," and its ends are tame. Until recently topologists did not know if "geometrically infinite" manifolds, where the spheres shrink down to an infinitely fine froth, also had tame ends. (© JosLeys.)*

Classifying Hyperbolic Manifolds— All's Well That Ends Well

Dana Mackenzie

WHILE THE STATUS of one famous unsolved problem in topology remained uncertain, topologists conquered four other major problems in 2004. The four theorems that went into this unique sundae—the Ending Laminations Conjecture, the Tame Ends Conjecture, the Ahlfors Measure Conjecture, and the Density Conjecture—were not household names, even among mathematicians. Yet to topologists who study three-dimensional spaces, they were as familiar (and went together as well) as chocolate, vanilla, hot fudge sauce and a maraschino cherry.

The Poincaré Conjecture (see "First of Seven Millennium Problems Nears Completion," p. 2) can be compared to the quest for the Loch Ness Monster: Whether you discover it or prove it doesn't exist, fame and glory are sure to follow. On the other hand, if topologists spent all their time hunting Loch Ness monsters, the subject would never advance. Someone has to study and classify the spaces that are already known. That is what the Ending Laminations Conjecture and Tame Ends Conjecture are all about. They complete the taxonomy of a very broad category of three-dimensional spaces, called hyperbolic 3-manifolds. Though they may not be quite as glamorous as deep-sea monsters, they are much more numerous.

"I've been working on the Poincaré Conjecture for most of my career," confesses Danny Calegari of Caltech. But when rumors started circulating in 2003 that Grisha Perelman had solved it, Calegari decided he had better look for a new problem to work on. Dave Gabai of Princeton University suggested that they could team up to work on the Tame Ends Conjecture, which Gabai had already been thinking about since 1996. It was a problem that had been bouncing around for thirty years, since it was first proposed by Al Marden of the University of Minnesota. At the time he proposed it, Marden says, "It was pie in the sky. No one had the vaguest idea how to prove it." Gabai had already tried to prove the Tame Ends Conjecture once, with Michael Freedman. Although their program had not worked—they found a counterexample to their original idea—Gabai wanted wanted help with a promising variant of Freedman's approach, called "shrinkwrapping."

Meanwhile, Ian Agol of the University of Illinois at Chicago was going through a very similar process. He, too, was frustrated with trying to understand Perelman's work on the Poincaré Conjecture, but had an idea how to prove the Tame

Danny Calegari. *(Photo courtesy of Danny Calegari.)*
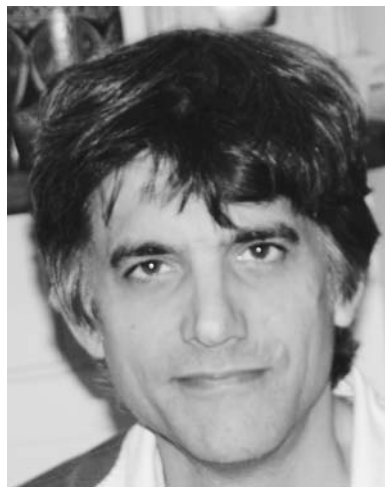
Dave Gabai.

Ian Agol. *(Photo courtesy of Ian Agol.)*

Jeffrey F. Brock. *(Photo courtesy of the University of Texas at Austin.)*

Ends Conjecture. Both groups—Gabai and Calegari together, and Agol working alone—finished their work at roughly the same time.

The proof of the Ending Laminations Conjecture, on the other hand, resulted from an intensive multi-year effort. The classification of hyperbolic manifolds depends on identifying a "feature" that topologists didn't even know about until William Thurston, then at Princeton University, discovered it in the 1970s. Thurston sketched out a rough idea for how this part, called an ending lamination, could be used to identify hyperbolic manifolds, but left huge gaps for other mathematicians to fill. The first step was a rigorous proof that ending laminations even existed—a proof that was provided by Francis Bonahon of the University of Southern California. The second step, using the laminations as a scaffolding to piece together a whole manifold, looked so difficult that only one mathematician seriously believed he could do it. Yair Minsky of Yale University began working on the Ending Laminations Conjecture shortly after receiving his Ph.D. from Princeton in 1989, and turned it into his career project. Working with a steady stream of collaborators—most notably Howard Masur, Dick Canary, and Jeff Brock—he gradually stitched together pieces of the proof until, by December 2004, he was finally ready to pronounce it complete. "In my mind," says Bonahon, "the real achievement is the work of Brock, Canary and Minsky. Agol, Calegari, and Gabai put a cherry on top—although a very impressive cherry." Minsky is somewhat more modest: "I'm glad that Francis feels this way, but personally I am happy with the role of one out of several cherries on a very nice sundae." Marden, on the other hand, dismisses all talk of sundaes and emphasizes the great importance of both theorems. "The most fundamental [result] is the tameness," he says. "For without knowing that, both ending laminations and density would be known only for 'tame' manifolds. There would be no way of knowing how encompassing a class this is." On the other hand, he adds, "The deepest of the theorems, the one that was hardest and required the deepest penetration into the hyperbolic geometry is ending laminations."

The final two ingredients in the sundae, the Density Conjecture and the Ahlfors Measure Conjecture, stem from an earlier era in the study of hyperbolic geometry, before Thurston revamped the subject and made it part of topology. In the 1960s, such mathematicians as Lars Ahlfors and Lipman Bers had taken a completely different approach to the subject, viewing it as a branch of complex analysis (the study of functions of complex numbers). They were interested in certain symmetries of complex functions known as Mobius transformations. Their original project was to classify groups of Mobius transformations, known as *Kleinian groups* (to be described below). But as often happens in mathematics, the problem could not be solved in the framework it was originally conceived in; it was only after Thurston rephrased the problem in terms of hyperbolic manifolds that major progress became possible. "We didn't even know how little we knew until Thurston," says Marden.

Ahlfors found that there were certain nicely behaved Kleinian groups that could be classified with the tools of complex analysis, but there were others that could not. The relation between them was very much like the relation between rational and irrational numbers. The decimal expansion of rational numbers marches in a precise cadence forever; irrational numbers may act rational for a long time, but they always break stride and follow their own drummer eventually. The Density Conjecture was a sort of formalization of this metaphor. It states that any hyperbolic manifold (or any group of Mobius transformations) can be approximated "arbitrarily well" by a well-behaved one.

What are hyperbolic manifolds, what are Mobius transformations, and what do they have to do with one another? A topologist would answer this question in one sentence: a hyperbolic manifold is a quotient of hyperbolic space by a discrete group of Mobius transformations. (He would mumble a few more words that sound like "orbifold" and "finitely generated," but non-topologists are free to ignore the mumbled part.) As usual in mathematics, quite a bit of explanation is needed to understand the one-sentence answer.

A three-dimensional manifold is the mathematician's way of describing the universe we live in. On a small scale, a manifold looks exactly like our familiar three-dimensional space, possibly with subtle distortions due to curvature. In places where the curvature is positive, parallel lines (which you can think of as light rays) bend toward each other as if they were passing through a lens; where the curvature is negative, light rays bend away from each other. The simplest kind of universe is one where the curvature is the same at every point. Such a manifold is called *hyperbolic* if the curvature is negative. Thurston was the first mathematician to realize that the vast majority of interesting three-dimensional manifolds are hyperbolic, or can be put together out of hyperbolic pieces.

On a large scale, manifolds can connect up in complicated ways, with wormholes and handles that may or may not exist in our universe. They may also have "ends," which are tubelike or flaring conelike structures that extend infinitely far into the distance. The Tame Ends Conjecture, as its name implies, states that the ends of hyperbolic manifolds cannot get very complicated. The Ending Laminations Conjecture complements this by stating that the geometry of tame ends completely controls the rest of the space. If you were the god of a hyperbolic universe with tame ends, you would find it very compliant. You would be able to change its shape just by tugging on its ends, and people living inside the universe would not have any say at all. (See Figure 1.)

An extreme case is a hyperbolic universe of finite size. Such a universe has no ends to tug on, and thus the universe can have only one possible shape. To be more precise, each possible *topology* has one unique *geometry* of constant negative curvature. That is exactly the content of the Mostow Rigidity Theorem, proved in 1973 by George Mostow, which Masur calls "the most influential piece of mathematics in geometry in the last 35 years." One of the main motivations behind the Tame Ends Conjecture and the Ending Laminations Conjecture was to explain how rigidity translates to hyperbolic universes that are infinitely large.

Yair Minsky. *(Photo courtesy of Yair Minsky.)*

Dick Canary. *(Photo courtesy of The University of Michigan Photo Services.)*
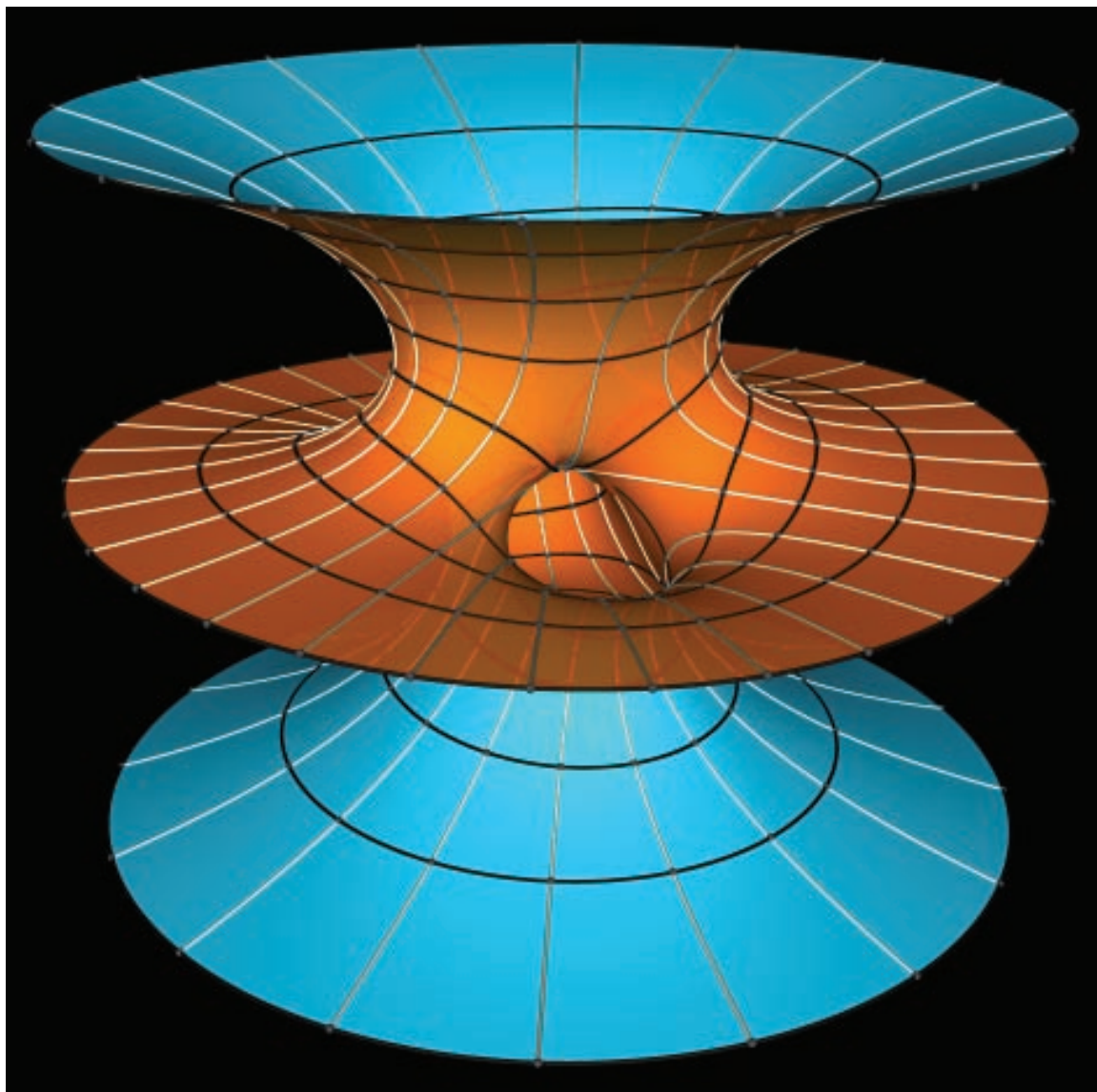
Figure 1. *An "end" of a non-compact manifold can be thought of as a piece of the manifold that stretches out to infinity. This image shows a famous surface, called the Costa-Hoffmann-Meeks surface, that has three ends—the three roughly planar pieces that stretch out to infinity. (Figure used with permission of Matthias Weber.)*

As implied by the one-line definition of hyperbolic manifolds given above, there is a second way of looking at them that doesn't involve universes, light rays, or benevolent gods. It involves Mobius transformations instead.

Most math majors first encounter Mobius transformations in a course on complex functions. In complex analysis, the Euclidean plane is represented by complex numbers $z = x + iy$, rather than by ordered pairs (x, y). Functions that map the plane to itself without distorting angles turn out to be particularly important and easy to represent with complex numbers; such functions are called *conformal*. The most general kind of conformal map that doesn't "miss" any points on the plane is called a Mobius transformation, and it has a particularly simple formula: $f(z) = (az + b)/(cz + d)$. (Here the variable

z and the constants *a, b, c, d* are all complex numbers.) Again, the purists would mumble words like "orientation-preserving" and "point at infinity," but you are free to ignore them unless you are being tested.

The wonder of Mobius transformations is that there is also a geometric or pictorial way of looking at them. You can create any Mobius transformation by doing a series of reflections in the complex plane, using either an ordinary flat mirror or a curved, circular one.

Once you have mirrors, you can put them together to create a "hall of mirrors" effect. In an ordinary room whose walls are flat mirrors, you can see copies of yourself in every direction you look. They form a crowd of clones (half of them mirror-images, the others exact copies of you) that seem to recede into the infinite distance. But what happens if you replace the flat mirrors with circular ones? Felix Klein, a nineteenth-century German geometer, discovered an amazing fact. You would still see infinitely many clones of yourself. But they would *not* appear to recede into the infinite distance. If there were three circular mirrors, the reflections would accumulate along a circle; and if there were four or more mirrors, the reflections would accumulate along a *limit set*, an extraordinarily jagged curve that Klein tried and failed to sketch. Nowadays we know why he found it so hard: It was the first appearance in the mathematical literature of a *fractal*.

In the computer age, we are no longer encumbered by the limitations of human imagination and draftsmanship. Pictures of fractals have become iconic images of the computer age, and pictures of Kleinian limit sets are among the most beautiful. A selection of images corresponding to different mirror sizes and placements is shown here. (Note that the circular mirrors have been replaced in some examples by spheres.) Every Kleinian limit set has its own unique style and beauty. Technically, the hall of mirrors is called a *Fuchsian group* (if the mirrors are circles in the plane) or a *Kleinian group* (if they are spheres in space). (See Figure 2.)

You can play around with Fuchsian groups and Kleinian groups to your heart's content without ever mentioning a single word of topology. However, the more you play, the more you will get the feeling that there is some organizing principle behind them. To understand that principle, you need topology and the concept of hyperbolic manifolds. (See Figure "Hall of Mirrors" on p. 14.)

As the one-line definition of hyperbolic manifolds said, you get a hyperbolic manifold by taking a quotient of the hyperbolic plane (or space) by a Fuchsian (or Kleinian) group. What does this mean?

Henri Poincaré—yes, the same man who came up with the Poincaré Conjecture—realized that if you are standing in a hall of mirrors, you have no way of telling whether you are in an infinite universe with infinite copies of yourself, or just a single room with mirrors on the walls. You might argue that there is one surefire if somewhat painful way to tell the difference: If you are in a single room, if you keep walking in one direction you will eventually bump into a wall. However, mathematically it is easy enough to arrange for the mirrors to be permeable, so that you can step right through them like Alice through the
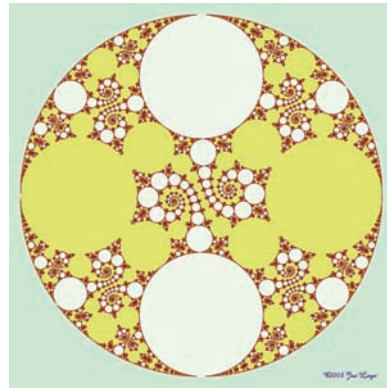
Figure 2. *This beautiful fractal portrays the action of a Kleinian group (a discrete group of Mobius transformations) on the complex plane. Any white disc can be mapped to any other white disc by one of the transformations in the Kleinian group. The group also maps the shaded region to itself. The fractal boundary between the two regions is the limit set of the group. The shaded region can also be thought of as the union of an infinite collection of "rooms" in a "hall of mirrors." An observer inside the shaded region cannot tell whether she is in a single room with mirrors or a gallery with infinitely many rooms. Even if she is in a single room, she will see infinitely many reflections of herself clustered in the distance, and the apparent shape traced out by these reflections will be identical to the Kleinian limit set. (©JosLeys.)*

Figure 3a. "Sea Turtles 1". *For caption, see Figure 3b on next page. (©JosLeys.)*

looking glass. If the universe is infinite, you would keep going into a new room that looks like the one you just left. If the universe is finite, you would re-emerge somewhere else in your original room. Either way, you cannot tell the difference. You are living in . . . the Quotient Zone. (Cue *Twilight Zone* music.)

A beautiful two-dimensional example of this phenomenon can be seen in the graphics of Jos Leys, a Belgian artist who is strongly inspired by the late M.C. Escher (see Figure 3). In his picture "Sea Turtles," we apparently see an entire plane with infinitely many turtles. Or do we? We could get exactly the same picture by placing mirrors along the backbones of four of the turtles, forming a square. In the Quotient Zone, there are only four "real" half-turtles, and everything else is a reflection of a reflection of a reflection. An example in hyperbolic geometry is provided by Leys' picture, "Fish." Now the backbones of six adjacent fish form a right-angled hexagon (a figure that exists
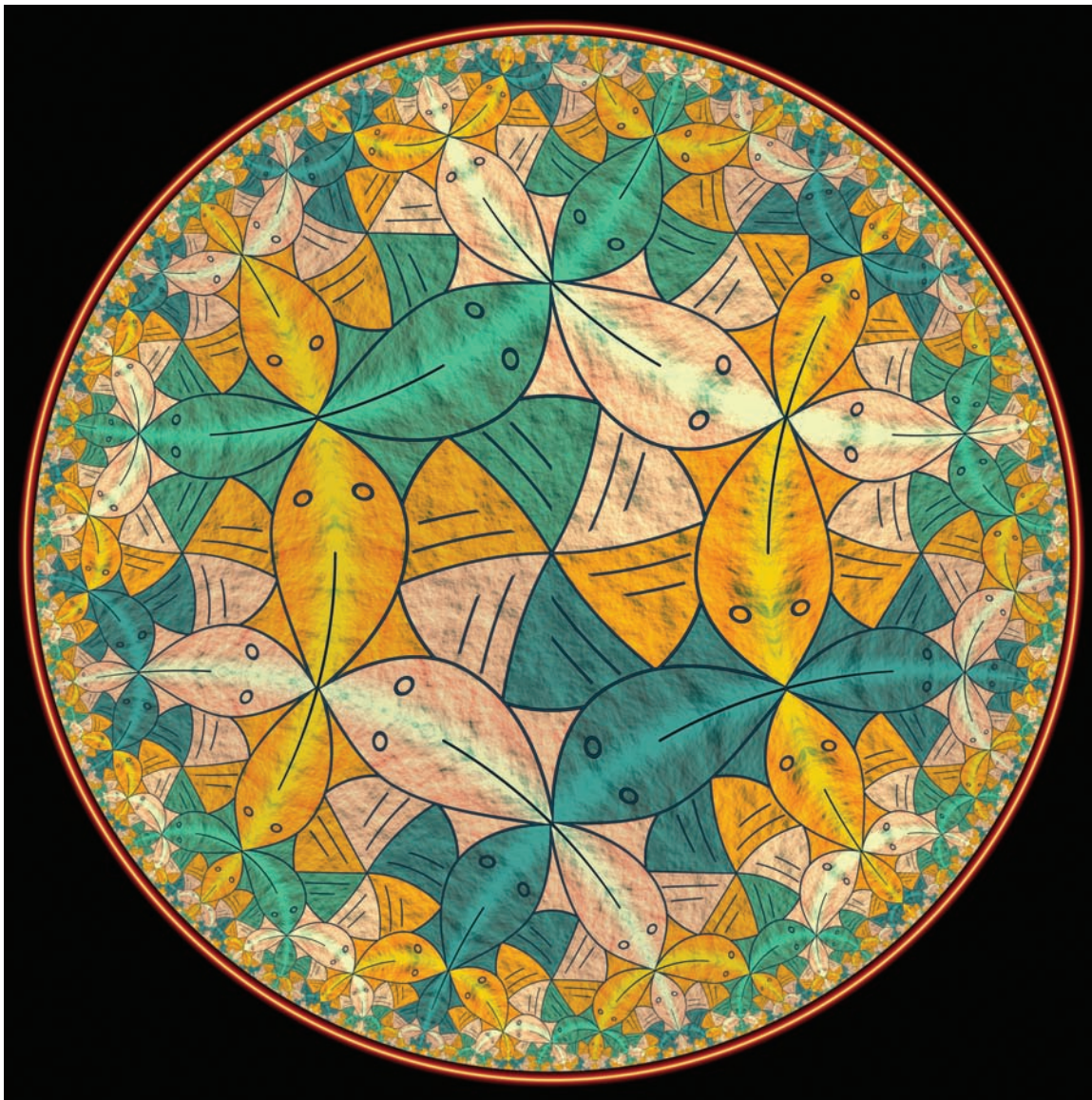
Figure 3b. "Fish". *Two tessellations by Jos Leys illustrate the difference between Euclidean and hyperbolic geometry. "Sea Turtles 1" is a tessellation of the Euclidean plane. Every turtle has the same size and shape. "Fish" is a tessellation of the hyperbolic plane. Every fish has the same size and shape, when measured according to the rules of hyperbolic geometry. In each image, the backbones of the animals reveal the underlying geometry of the space they live in. The backbones of the turtles form a square grid, while the backbones of the fish form a pattern of right-angled hexagons. Right-angled hexagons exist only in hyperbolic geometry, just as squares exist only in Euclidean geometry. (© JosLeys.)*

only in hyperbolic geometry). Are there infinitely many fish, or only six half-fish? I'm not telling. Sometimes it is more convenient to look at it one way, and sometimes it is more convenient to look at it the other.

   The first way is better if you want to visualize the ends of a hyperbolic manifold. In this view, identical copies of the manifold fill an entire disk (in two dimensions) or ball (in three dimensions), just as Leys' pictures are filled with turtles and fish. The disk or ball is called the *Poincare disk model* of the *hyper-*

bolic plane (or *hyperbolic space*). Ends arise if the hyperbolic manifold (or, equivalently, any one room of the hall of mirrors) extends all the way out to the boundary of the Poincaré disk. This makes the manifold infinitely large because of the way that distances are measured in hyperbolic geometry. The center of the Poincaré disk is infinitely far from its boundary, as you can see from looking at Leys' picture. Each of the fishes in the picture is defined to be the same size in hyperbolic geometry, and you have to pass by infinitely many fishes to reach the boundary. In other words, you have to travel an infinite distance.

So far there has not been much of a difference between 2- and 3-dimensional hyperbolic manifolds, aside from the sub-
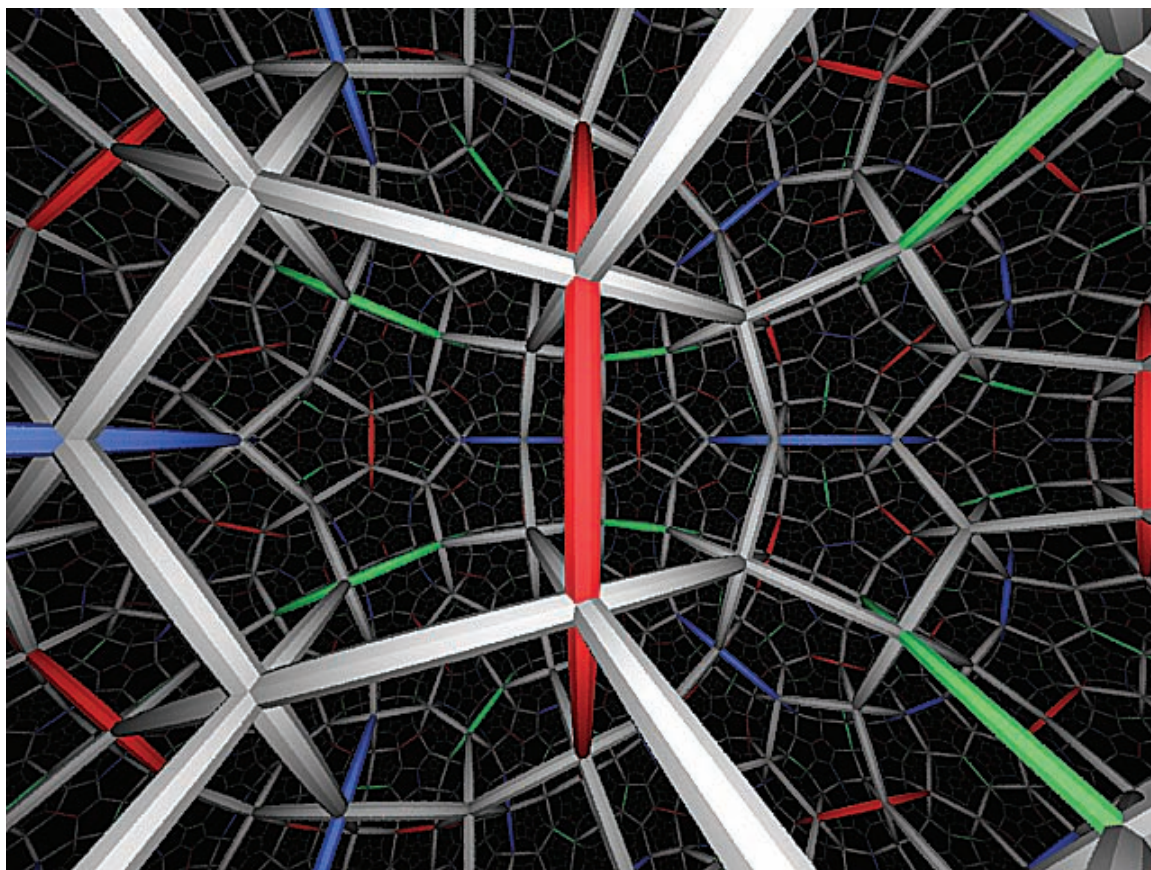
Figure 4. *The view from "inside" a hyperbolic 3-manifold. The manifold has finite volume and size, but it appears infinite for the same reason that a room with mirrors does. The adjacent "rooms" in this picture are really the same room, apparently rotated and translated. In this image the manifold is compact (i.e., the rooms have finite size), but the same methods can be used to study non-compact 3-manifolds as well. (Figure courtesy of The Geometry Center, University of Minnesota, © 1990. All rights reserved.)*

stitution of "space" for "plane" and "ball" for "disk." As in the 2-dimensional case, 3-dimensional manifolds can be pictured as individual rooms in a hall of mirrors, with the various reflections of the room apparently filling out all of hyperbolic space (see Figure 4). However, when we start talking about ends, a dramatic difference appears. In 2 dimensions, the ends are just arcs on the boundary circle, and the limit set of the Fuchsian group is the collection of endpoints of those arcs. But in 3 dimensions, we have a boundary *sphere*. The hyperbolic manifold corresponds to a "room" inside the sphere, with exotic matching rules on the walls. (Remember that when you are in the Quotient Zone, walking through one wall causes you to reemerge from a different wall in the same room; the matching rules say which one.) The ends are polygons on the boundary sphere, which have matching rules of their own. These rules glue the edges together in such a way that the polygons become 2-dimensional surfaces (such as doughnuts or pretzels). The Kleinian limit set is a gorgeous fractal, where infinitely many of the end-polygons cluster together. The limit set and the ends are complementary; everything on the boundary sphere that is not part of an end of the hyperbolic manifold is part of the Kleinian limit set. (See Hall of Mirrors, p. 14.)

Back in the 1960s, Ahlfors showed that if the number of walls of any "room" is finite, then the shape of the end polygons translates directly to a unique hyperbolic geometry on the "rooms" and the hyperbolic manifold they represent. In fact, according to Marden, Ahlfors initially thought that his theorem settled the question once and for all. Topologists now call it the "geometrically finite" case. But it turned out that in three dimensions (unlike two) the "rooms" of the hall of mirrors could contain infinitely many sides. In this case the Kleinian limit set suddenly changes its nature from an elegant collection of filigrees into a violently jagged, tentacled monster that gobbles up almost the entire boundary sphere. (This sudden transformation is the essence of the Ahlfors Measure Conjecture, the fourth theorem in the "sundae.") Even with a computer, it is hard to draw an accurate picture of it. It was this monster—the case of the "geometrically infinite" ends—that required the efforts of six people (Agol, Calegari, Gabai, Brock, Canary and Minsky) to tame and assign its proper taxonomy.

At this point the saga of the hyperbolic manifolds splits, like the *Lord of the Rings* saga, into two parties. Agol, Calegari and Gabai chose to work on the taming of the ends.

In the geometrically infinite case, the ends of the hyperbolic manifold actually do not make it to the boundary sphere. The reason is that the Kleinian limit set takes away so much of the sphere that there is nothing left for the ends. Unfortunately, this completely negates the usefulness of the first way of looking at Kleinian groups: we can't very well describe how the ends control the geometry of the hyperbolic manifold if we can't even find the ends. This is, perhaps, why Ahlfors was stymied by the geometrically infinite case.

However, the ends are still there, and you can still study them by going to the Quotient Zone point of view. In this interpretation, the hyperbolic manifold (the "room" in the hall of mirrors) has a long, undulating tunnel that reaches out toward infinity.

At this point the saga of the hyperbolic manifolds splits, like the *Lord of the Rings* saga, into two parties.

The end of the manifold is *tame* if it is topologically the same as a straight tunnel with no undulations. The reason the ends are hard to tame is that the tunnel may twist infinitely often, and in the course of this twisting become so knotted that it can no longer be straightened out. Somehow the constant negative curvature of the hyperbolic manifold (remember that?) must prevent this kind of infinite twist. The proof by Calegari and Gabai involved a technique they invented, called "shrinkwrapping." They showed that you could take a short loop that circles the tunnel and "shrinkwrap" it—that is, you push a surface outward from the "core" of the manifold until it gets caught on the loop, then pull it tight. A simple but elegant geometric argument (which works only because of the manifold's constant negative curvature) shows that the area of the shrinkwrap does not increase as the loop moves farther and farther out along the tunnel. But that means that the end cannot be infinitely twisted and knotted because if it were, the shrinkwrap for the more distant loops would get tangled up with the closer loops, and the areas would have gotten bigger and bigger. Therefore, the end must have been tame to begin with!

Meanwhile, Minsky and his troops were working to understand the structure of ends once they had been tamed. Once you know the end has the structure of a straight tunnel, you can take any cross-section of that (three-dimensional) tunnel and get a two-dimensional surface—for example, a torus with a puncture in it. Every cross- section will be the same, so it is possible to pick one cross-section as a model for all the others.

Next, Minsky (following Thurston's idea) considered a sequence of loops heading out the end of the tube—the same loops that Calegari and Gabai used in their shrinkwrapping argument. He slid these loops back to the reference surface. But when you slide a curve in hyperbolic geometry, it invariably becomes longer. (This is because of the defining property of negative curvature: parallel lines diverge. Hence the ends of a line segment get farther and farther apart when you slide them in parallel directions.) Thus if your first curve wraps around the reference surface once, the second one will be longer and it will have to wrap around it more than once—or perhaps once longitudinally and once latitudinally, creating a "barber pole" effect.

As the curves get longer and longer, one of two things could happen. They could start to retrace the same path (an easy way to increase distance—just travel the same path over and over), or they could fail to close up ever, and just create a denser and denser sequence of barber pole stripes. In the limit, the barber pole stripes become infinitely long and infinitely dense—and that limit is called a *lamination* of the reference surface. It was this structure, Thurston believed, that contained all the relevant information about the shape of the end, and completely determined the geometry of the rest of the manifold. This assertion became known as the Ending Laminations Conjecture.

One challenge for Minsky was to figure out how different ends communicate with each other—because if there is more than one end, each of them will play a role in shaping the manifold. The simplest, and first, case that he worked on was that of a manifold with two ends, each with the same reference surface. For example, if the reference surface were a punctured

torus, the manifold would be a thick rubber inner tube with a nail hole running from the outside surface to the inside surface. One might call it a punctured tire. The outside of the inner tube would have one ending lamination and the inside, which you cannot see, would have another.
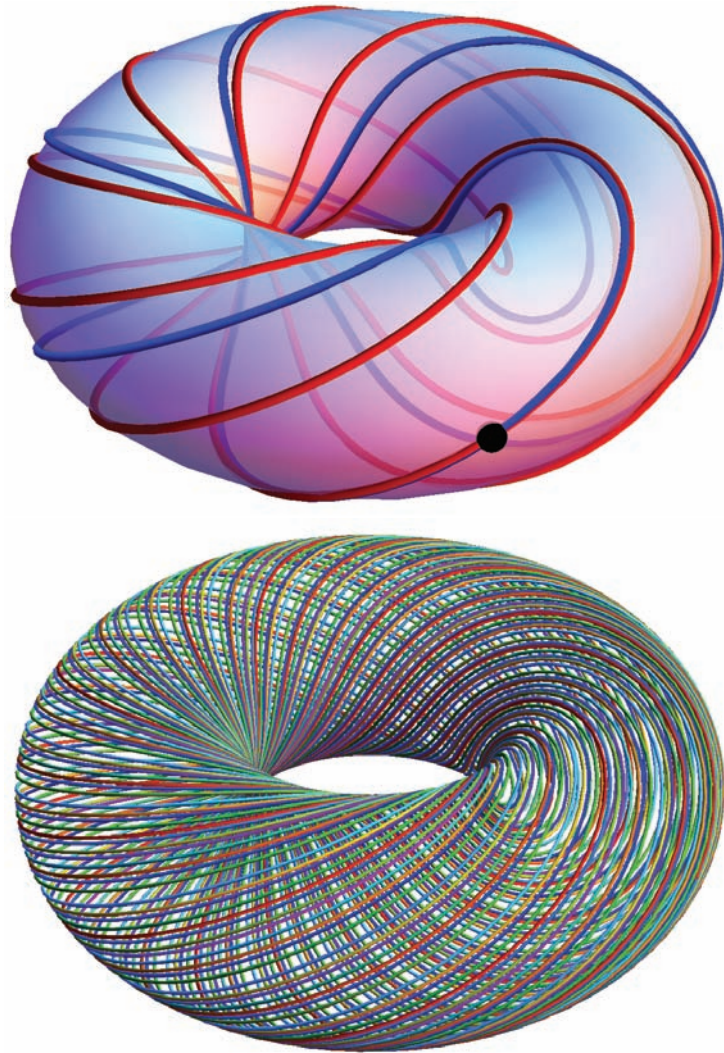
In this example, a barber pole stripe that winds around the reference surface (the punctured torus) p times longitudinally and q times latitudinally corresponds to a rational number, p/q. Ending laminations correspond to irrational numbers, such as the golden ratio (1.618033985...). To move from one ending lamination to another, one proceeds by a series of hops from one "nearby" rational number to the next. In the case of the golden ratio, one might hop along Fibonacci ratios—from 13/8 (1.625) to 8/5 (1.6) to 5/3 (1.666...) to 3/2 (1.5) and so on, gradually moving away from the golden ratio and towards the irrational number representing the second ending lamination. (See Figure 5, next page.)

At each step, hopping from one ratio to the next corresponds to moving from one barbershop spiral to another one that does not intersect it. (They only intersect at the puncture point, which has been removed for precisely that reason.) It is hard for two different spirals not to intersect, and so the information on which spirals do not intersect turns out to be crucial in reconstructing the geometry of the manifold. Minsky showed (in about 1994) that if you go all the way from one lamination to another (which requires an infinite number of "hops") you will get all the information you need to build a complete model of your manifold. At first it is not a perfect model, but sort of a stitched-together Frankenstein-like version of it. But in their 120-page paper, released as a preprint in 2004, he showed with Brock and Canary how to smooth out the stitching so that you end up with an exact copy of your original manifold, with the desired ending laminations on both ends. Most impressively, they described how to do it for all manifolds, not just the punctured-tire manifold described here. Thus, he concluded, Thurston was right. The ending laminations for all geometrically infinite ends, plus the shape of any geometrically finite ends, completely describe the geometry of any hyperbolic manifold. (One more time there are some mumbled words that you can ignore, concerning a "finitely generated fundamental group.")

What can you do for an encore after bringing a 15-year research program to a close? "In this field we've been obsessed with manifolds of infinite volume for a long time," says Brock. "One thing that we're doing now is taking the technology of these proofs and translating it back to manifolds of finite volume."

For example, Brock says that any three-dimensional hyperbolic manifold can be sliced into two pieces in a special way called a *Heegaard splitting*. The two pieces are topologically identical, and they look like balls with lots of handles of different shapes attached. The trouble with the Heegaard splitting is that the pieces have open, raw edges—the places where the original manifold was cut open. Topologists did not have the tools to relate combinatorial information about the edges to the geometry of the pieces. But now they do. Manifolds with open edges are a lot like manifolds with ends (although the edges

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What can you do for an encore after bringing a 15-year research program to a close?

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Figure 5. *A lamination on a torus is obtained as a limit of torus knots, i.e., curves that wind around the torus and close up. Here, the blue curve circumnavigates the torus 3 times in latitude while going around 5 times in longitude. Thus it has "slope" 5/3 = 1.666. ... Similarly, the red curve has "slope" 8/5 = 1.6. These two curves intersect only at the puncture point (black dot). The multicolored curve has an irrational "slope" of 1.618. ... It winds around forever without ever closing up or intersecting itself. This is called a lamination. (Graphics created by Michael Trott using Mathematica.)*

do not extend out to infinity). The region around the edges can be equipped with a curve complex, the very same type of scaffolding that Minsky used to generate a blueprint of a manifold with ends. "This is neat because there are people who understand the topology of Heegaard splittings in a very deep way," says Brock. "Our challenge is to relate this topology to the curve complex."

Minsky also is optimistic that there is a lot still to do. The fact that you have a description of a manifold's shape does not mean you have a *convenient* description. An analogous situation can be found even in high-school Euclidean geometry. The three side lengths of a triangle "completely describe" its shape—but that doesn't necessarily make it easy to answer concrete questions about the triangle. For example, geometers still do not know which triangles have periodic billiard-ball trajectories—even though that information must somehow be encoded in the side lengths. Minsky says that topologists are still a long way from understanding how the ending laminations, which "completely describe" a hyperbolic manifold, relate to other geometric properties of the manifold.

As for Marden, he is thrilled to see his conjecture proved after thirty years. "So much has happened," he says. "If you ever questioned whether there is progress being made in mathematics, this is a very clear case. These proofs could not have been done thirty years ago."

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