The Ultimate Companion to
A Comprehensive Course in Analysis

A five-volume reference set by
Barry Simon

This booklet includes:

- Preface to the Series
- Tables of Contents and Prefaces (Parts 1, 2A, 2B, 3, and 4)
- Sample Section: Classical Fourier Series (Section 3.5 from Part 1)
- Subject Index
- Author Index
- Combined Index of Capsule Biographies

AMS
American Mathematical Society

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Providence, Rhode Island

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Preface to the Series

> Young men should prove theorems, old men should write books.
>
> —*Freeman Dyson*, quoting G. H. Hardy[1]

Reed–Simon[2] starts with "Mathematics has its roots in numerology, geometry, and physics." This puts into context the division of mathematics into algebra, geometry/topology, and analysis. There are, of course, other areas of mathematics, and a division between parts of mathematics can be artificial. But almost universally, we require our graduate students to take courses in these three areas.

This five-volume series began and, to some extent, remains a set of texts for a basic graduate analysis course. In part it reflects Caltech's three-terms-per-year schedule and the actual courses I've taught in the past. Much of the contents of Parts 1 and 2 (Part 2 is in two volumes, Part 2A and Part 2B) are common to virtually all such courses: point set topology, measure spaces, Hilbert and Banach spaces, distribution theory, and the Fourier transform, complex analysis including the Riemann mapping and Hadamard product theorems. Parts 3 and 4 are made up of material that you'll find in some, but not all, courses—on the one hand, Part 3 on maximal functions and H^p-spaces; on the other hand, Part 4 on the spectral theorem for bounded self-adjoint operators on a Hilbert space and det and trace, again for Hilbert space operators. Parts 3 and 4 reflect the two halves of the third term of Caltech's course.

[1] Interview with D. J. Albers, The College Mathematics Journal, **25**, no. 1, January 1994.

[2] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press, New York, 1972.

While there is, of course, overlap between these books and other texts, there are some places where we differ, at least from many:

(a) By having a unified approach to both real and complex analysis, we are able to use notions like contour integrals as Stietljes integrals that cross the barrier.
(b) We include some topics that are not standard, although I am surprised they are not. For example, while discussing maximal functions, I present Garcia's proof of the maximal (and so, Birkhoff) ergodic theorem.
(c) These books are written to be keepers—the idea is that, for many students, this may be the last analysis course they take, so I've tried to write in a way that these books will be useful as a reference. For this reason, I've included "bonus" chapters and sections—material that I do not expect to be included in the course. This has several advantages. First, in a slightly longer course, the instructor has an option of extra topics to include. Second, there is some flexibility—for an instructor who can't imagine a complex analysis course without a proof of the prime number theorem, it is possible to replace all or part of the (nonbonus) chapter on elliptic functions with the last four sections of the bonus chapter on analytic number theory. Third, it is certainly possible to take all the material in, say, Part 2, to turn it into a two-term course. Most importantly, the bonus material is there for the reader to peruse long after the formal course is over.
(d) I have long collected "best" proofs and over the years learned a number of ones that are not the standard textbook proofs. In this regard, modern technology has been a boon. Thanks to Google books and the Caltech library, I've been able to discover some proofs that I hadn't learned before. Examples of things that I'm especially fond of are Bernstein polynomials to get the classical Weierstrass approximation theorem, von Neumann's proof of the Lebesgue decomposition and Radon–Nikodym theorems, the Hermite expansion treatment of Fourier transform, Landau's proof of the Hadamard factorization theorem, Wielandt's theorem on the functional equation for $\Gamma(z)$, and Newman's proof of the prime number theorem. Each of these appears in at least some monographs, but they are not nearly as widespread as they deserve to be.
(e) I've tried to distinguish between central results and interesting asides and to indicate when an interesting aside is going to come up again later. In particular, all chapters, except those on preliminaries, have a listing of "Big Notions and Theorems" at their start. I wish that this attempt to differentiate between the essential and the less essential

didn't make this book different, but alas, too many texts are monotone listings of theorems and proofs.

(f) I've included copious "Notes and Historical Remarks" at the end of each section. These notes illuminate and extend, and they (and the Problems) allow us to cover more material than would otherwise be possible. The history is there to enliven the discussion and to emphasize to students that mathematicians are real people and that "may you live in interesting times" is truly a curse. Any discussion of the history of real analysis is depressing because of the number of lives ended by the Nazis. Any discussion of nineteenth-century mathematics makes one appreciate medical progress, contemplating Abel, Riemann, and Stieltjes. I feel knowing that Picard was Hermite's son-in-law spices up the study of his theorem.

On the subject of history, there are three cautions. First, I am not a professional historian and almost none of the history discussed here is based on original sources. I have relied at times—horrors!—on information on the Internet. I have tried for accuracy but I'm sure there are errors, some that would make a real historian wince.

A second caution concerns looking at the history assuming the mathematics we now know. Especially when concepts are new, they may be poorly understood or viewed from a perspective quite different from the one here. Looking at the wonderful history of nineteenth-century complex analysis by Bottazzini–Grey[3] will illustrate this more clearly than these brief notes can.

The third caution concerns naming theorems. Here, the reader needs to bear in mind Arnol'd's principle:[4] *If a notion bears a personal name, then that name is not the name of the discoverer* (and the related Berry principle: *The Arnol'd principle is applicable to itself*). To see the applicability of Berry's principle, I note that in the wider world, Arnol'd's principle is called "Stigler's law of eponymy." Stigler[5] named this in 1980, pointing out it was really discovered by Merton. In 1972, Kennedy[6] named Boyer's law *Mathematical formulas and theorems are usually not named after their original discoverers* after Boyer's book.[7] Already in 1956, Newman[8] quoted the early twentieth-century philosopher and logician A. N. Whitehead as saying: "Everything of importance has been said before by somebody who

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[3] U. Bottazzini and J. Gray, *Hidden Harmony—Geometric Fantasies. The Rise of Complex Function Theory*, Springer, New York, 2013.

[4] V. I. Arnol'd, *On teaching mathematics*, available online at `http://pauli.uni-muenster.de/~munsteg/arnold.html`.

[5] S. M. Stigler, *Stigler's law of eponymy*, Trans. New York Acad. Sci. **39** (1980), 147–158.

[6] H. C. Kennedy, *Classroom notes: Who discovered Boyer's law?*, Amer. Math. Monthly **79** (1972), 66–67.

[7] C. B. Boyer, *A History of Mathematics*, Wiley, New York, 1968.

[8] J. R. Newman, *The World of Mathematics*, Simon & Schuster, New York, 1956.

did not discover it." The main reason to give a name to a theorem is to have a convenient way to refer to that theorem. I usually try to follow common usage (even when I know Arnol'd's principle applies).

I have resisted the temptation of some text writers to rename things to set the record straight. For example, there is a small group who have attempted to replace "WKB approximation" by "Liouville–Green approximation", with valid historical justification (see the Notes to Section 15.5 of Part 2B). But if I gave a talk and said I was about to use the Liouville–Green approximation, I'd get blank stares from many who would instantly know what I meant by the WKB approximation. And, of course, those who try to change the name also know what WKB is! Names are mainly for shorthand, not history.

These books have a wide variety of problems, in line with a multiplicity of uses. The serious reader should at least skim them since there is often interesting supplementary material covered there.

Similarly, these books have a much larger bibliography than is standard, partly because of the historical references (many of which are available online and a pleasure to read) and partly because the Notes introduce lots of peripheral topics and places for further reading. But the reader shouldn't consider for a moment that these are intended to be comprehensive—that would be impossible in a subject as broad as that considered in these volumes.

These books differ from many modern texts by focusing a little more on special functions than is standard. In much of the nineteenth century, the theory of special functions was considered a central pillar of analysis. They are now out of favor—too much so—although one can see some signs of the pendulum swinging back. They are still mainly peripheral but appear often in Part 2 and a few times in Parts 1, 3, and 4.

These books are intended for a second course in analysis, but in most places, it is really previous exposure being helpful rather than required. Beyond the basic calculus, the one topic that the reader is expected to have seen is metric space theory and the construction of the reals as completion of the rationals (or by some other means, such as Dedekind cuts).

Initially, I picked "A Course in Analysis" as the title for this series as an homage to Goursat's *Cours d'Analyse*,[9] a classic text (also translated into English) of the early twentieth century (a literal translation would be

[9]E. Goursat, *A Course in Mathematical Analysis: Vol. 1: Derivatives and Differentials, Definite Integrals, Expansion in Series, Applications to Geometry. Vol. 2, Part 1: Functions of a Complex Variable. Vol. 2, Part 2: Differential Equations. Vol. 3, Part 1: Variation of Solutions. Partial Differential Equations of the Second Order. Vol. 3, Part 2: Integral Equations. Calculus of Variations*, Dover Publications, New York, 1959 and 1964; French original, 1905.

"of Analysis" but "in" sounds better). As I studied the history, I learned that this was a standard French title, especially associated with École Polytechnique. There are nineteenth-century versions by Cauchy and Jordan and twentieth-century versions by de la Vallée Poussin and Choquet. So this is a well-used title. The publisher suggested adding "Comprehensive", which seems appropriate.

It is a pleasure to thank many people who helped improve these texts. About 80% was TeXed by my superb secretary of almost 25 years, Cherie Galvez. Cherie was an extraordinary person—the secret weapon to my productivity. Not only was she technically strong and able to keep my tasks organized but also her people skills made coping with bureaucracy of all kinds easier. She managed to wind up a confidant and counselor for many of Caltech's mathematics students. Unfortunately, in May 2012, she was diagnosed with lung cancer, which she and chemotherapy valiantly fought. In July 2013, she passed away. I am dedicating these books to her memory.

During the second half of the preparation of this series of books, we also lost Arthur Wightman and Ed Nelson. Arthur was my advisor and was responsible for the topic of my first major paper—perturbation theory for the anharmonic oscillator. Ed had an enormous influence on me, both via the techniques I use and in how I approach being a mathematician. In particular, he taught me all about closed quadratic forms, motivating the methodology of my thesis. I am also dedicating these works to their memory.

After Cherie entered hospice, Sergei Gel'fand, the AMS publisher, helped me find Alice Peters to complete the TeXing of the manuscript. Her experience in mathematical publishing (she is the "A" of A K Peters Publishing) meant she did much more, for which I am grateful.

This set of books has about 150 figures which I think considerably add to their usefulness. About half were produced by Mamikon Mnatsakanian, a talented astrophysicist and wizard with Adobe Illustrator. The other half, mainly function plots, were produced by my former Ph.D. student and teacher extraordinaire Mihai Stoiciu (used with permission) using Mathematica. There are a few additional figures from Wikipedia (mainly under WikiCommons license) and a hyperbolic tiling of Douglas Dunham, used with permission. I appreciate the help I got with these figures.

Over the five-year period that I wrote this book and, in particular, during its beta-testing as a text in over a half-dozen institutions, I received feedback and corrections from many people. In particular, I should like to thank (with apologies to those who were inadvertently left off): Tom Alberts, Michael Barany, Jacob Christiansen, Percy Deift, Tal Einav, German Enciso, Alexander Eremenko, Rupert Frank, Fritz Gesztesy, Jeremy Gray, Leonard Gross, Chris Heil, Mourad Ismail, Svetlana Jitomirskaya, Bill Johnson, Rowan Killip, John Klauder, Seung Yeop Lee, Milivoje Lukic, Andre Martinez-Finkelshtein, Chris Marx, Alex Poltoratski, Eric Rains, Lorenzo Sadun, Ed Saff, Misha Sodin, Dan Stroock, Benji Weiss, Valentin Zagrebnov, and Maxim Zinchenko.

Much of these books was written at the tables of the Hebrew University Mathematics Library. I'd like to thank Yoram Last for his invitation and Naavah Levin for the hospitality of the library and for her invaluable help.

This series has a Facebook page. I welcome feedback, questions, and comments. The page is at `www.facebook.com/simon.analysis` .

Even if these books have later editions, I will try to keep theorem and equation numbers constant in case readers use them in their papers.

Finally, analysis is a wonderful and beautiful subject. I hope the reader has as much fun using these books as I had writing them.

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Preface to Part 1

> I warn you in advance that all the principles ... that I'll now tell you about, are a little false. Counterexamples can be found to each one—but as directional guides the principles still serve a useful purpose.
>
> —*Paul Halmos*[1]

Analysis is the infinitesimal calculus writ large. Calculus as taught to most high school students and college freshmen is the subject as it existed about 1750—I've no doubt that Euler could have gotten a perfect score on the Calculus BC advanced placement exam. Even "rigorous" calculus courses that talk about ε-δ proofs and the intermediate value theorem only bring the subject up to about 1890 after the impact of Cauchy and Weierstrass on real variable calculus was felt.

This volume can be thought of as the infinitesimal calculus of the twentieth century. From that point of view, the key chapters are Chapter 4, which covers measure theory—the consummate integral calculus—and the first part of Chapter 6 on distribution theory—the ultimate differential calculus.

But from another point of view, this volume is about the triumph of abstraction. Abstraction is such a central part of modern mathematics that one forgets that it wasn't until Fréchet's 1906 thesis that sets of points with no a priori underlying structure (not assumed points in or functions on $\mathbb{R}^n$) are considered and given a structure a posteriori (Fréchet first defined abstract metric spaces). And after its success in analysis, abstraction took over significant parts of algebra, geometry, topology, and logic.

[1] L. Gillman, P. R. Halmos, H. Flanders, and B. Shube, *Four Panel Talks on Publishing*, Amer. Math. Monthly **82** (1975), 13–21.

Abstract spaces are a distinct thread here, starting with topological spaces in Chapter 2, Banach spaces in Chapter 5 (and its special case, Hilbert spaces, in Chapter 3), and locally convex spaces in the later parts of Chapters 5 and 6 and in Chapter 9.

Of course, abstract spaces occur to set up the language we need for measure theory (which we do initially on compact Hausdorff spaces and where we use Banach lattices as a tool) and for distributions which are defined as duals of some locally convex spaces.

Besides the main threads of measure theory, distributions, and abstract spaces, several leitmotifs can be seen: Fourier analysis (Sections 3.5, 6.2, and 6.4–6.6 are a minicourse), probability (Bonus Chapter 7 has the basics, but it is implicit in much of the basic measure theory), convexity (a key notion in Chapter 5), and at least bits and pieces of the theory of ordinary and partial differential equations.

The role of pivotal figures in real analysis is somewhat different from complex analysis, where three figures—Cauchy, Riemann, and Weierstrass—dominated not only in introducing the key concepts, but many of the most significant theorems. Of course, Lebesgue and Schwartz invented measure theory and distributions, respectively, but after ten years, Lebesgue moved on mainly to pedagogy and Hörmander did much more to cement the theory of distributions than Schwartz. On the abstract side, F. Riesz was a key figure for the 30 years following 1906, with important results well into his fifties, but he doesn't rise to the dominance of the complex analytic three.

In understanding one part of the rather distinct tone of some of this volume, the reader needs to bear in mind "Simon's three kvetches":[2]

1. Every interesting topological space is a metric space.
2. Every interesting Banach space is separable.
3. Every interesting real-valued function is Baire/Borel measurable.

Of course, the principles are well-described by the Halmos quote at the start—they aren't completely true but capture important ideas for the reader to bear in mind. As a mathematician, I cringe at using the phrase "not completely true." I was in a seminar whose audience included Ed Nelson, one of my teachers. When the speaker said the proof he was giving was almost rigorous, Ed said: "To say something is almost rigorous makes as much sense as saying a woman is almost pregnant." On the other hand, Neils Bohr, the founding father of quantum mechanics, said: "It is the hallmark of any deep truth that its negation is also a deep truth."[3]

---

[2] `http://www.merriam-webster.com/dictionary/kvetch`

[3] Quoted by Max Delbruck, *Mind from Matter? An Essay on Evolutionary Epistemology*, Blackwell Scientific Publications, Palo Alto, CA, 1986; page 167.

We'll see that weak topologies on infinite-dimensional Banach spaces are never metrizable (see Theorem 5.7.2) nor is the natural topology on $C_0^\infty(\mathbb{R}^\nu)$ (see Theorem 9.1.5), so Kvetch 1 has counterexamples, but neither case is so far from metrizable: If X^* is separable, the weak topology restricted to the unit ball of X is metrizable (see Theorem 5.7.2). While $C_0^\infty(\mathbb{R}^\nu)$ is not metrizable, that is because we allow ordinary distributions of arbitrary growth. If we restrict ourselves to distributions of any growth restriction, the test function space will be metrizable (see Sections 6.1 and 6.2). But the real point of Kvetch 1 is that the reason for studying topological spaces is *not* (merely) to be able to discuss nonmetrizable spaces—it is because metrics have more structure than is needed—$(0,1)$ is not complete with its usual metric while $\mathbb{R}$ is, but they are the same as topological spaces. Topological spaces provide the proper language for parts of analysis.

$L^\infty([0,1],dx)$ and $\mathcal{L}(\mathcal{H})$, the bounded operators on a Hilbert space, $\mathcal{H}$, are two very interesting spaces which are *not* separable, so Kvetch 2 isn't strictly true. But again, there is a point to Kvetch 2. In many cases, the most important members of a class of spaces are separable and one has to do considerable gymnastics in the general case, which is never, or at most very rarely, used. Of course, the gymnastics can be fun, but they don't belong in a first course. We illustrate this by including separability as an axiom for Hilbert spaces. Von Neumann did also in his initial work, but over the years, this has been dropped in most books. We choose to avoid the complications and mainly restrict ourselves to the separable case.

Two caveats: First, the consideration of the nonseparable case can provide more elegant proofs! For example, the projection lemma of Theorem 3.2.3 was proven initially for the separable case using a variant of Gram–Schmidt. The elegant proof we use that exploits convex minimization was only discovered because of a need to handle the nonseparable case. Second, we abuse the English language. A "red book" is a "book." We include separability and complex field in our definition of Hilbert space. We'll use the terms "nonseparable Hilbert space" and "real Hilbert space," which are not Hilbert spaces!

In one sense, Kvetch 3 isn't true, but except for one caveat, it is. Every set, A, has its characteristic function associated with it. If the only interesting functions are Borel functions, the only interesting sets are Borel sets. While it is a more advanced topic that we won't consider, there are sets constructed from Borel sets, called analytic sets and Souslin sets which may not be Borel.[4] The kvetch is there to eliminate Lebesgue measurable sets and functions, that is, sets $A = B \triangle C$, where B is Borel, and $C \subset D$, a Borel set of Lebesgue measure zero. The end of Section 4.3 discusses why it is not

[4] See, e.g., V. Bogachev, *Measure Theory*, Springer, 2007.

a good idea to consider such sets (and functions) even though many books do and it's what the Carathéodory construction of Section 8.1 leads to.

The last issue we mention in this preface is that our approach to measure theory is different from the standard one—it follows an approach in the appendix of Lax[5] that starts with a positive functional, ℓ, on $C(X)$, completes $C(X)$ in the $\ell(|f|)$-norm, and shows that the elements of the completion are equivalence classes of Borel functions. For those who prefer more traditional approaches, Section 4.13 discusses general measure spaces and Section 8.1 discusses the Carathéodory outer measure construction.

[5] P. Lax, *Functional Analysis*, Wiley, 2002.

Preface to Part 2

Part 2 of this five-volume series is devoted to complex analysis. We've split Part 2 into two pieces (Part 2A and Part 2B), partly because of the total length of the current material, but also because of the fact that we've left out several topics and so Part 2B has some room for expansion. To indicate the view that these two volumes are two halves of one part, chapter numbers are cumulative. Chapters 1–11 are in Part 2A, and Part 2B starts with Chapter 12.

The flavor of Part 2 is quite different from Part 1—abstract spaces are less central (although hardly absent)—the content is more classical and more geometrical. The classical flavor is understandable. Most of the material in this part dates from 1820–1895, while Parts 1, 3, and 4 largely date from 1885–1940.

While real analysis has important figures, especially F. Riesz, it is hard to single out a small number of "fathers." On the other hand, it is clear that the founding fathers of complex analysis are Cauchy, Weierstrass, and Riemann. It is useful to associate each of these three with separate threads which weave together to the amazing tapestry of this volume. While useful, it is a bit of an exaggeration in that one can identify some of the other threads in the work of each of them. That said, they clearly did have distinct focuses, and it is useful to separate the three points of view.

To Cauchy, the central aspect is the differential and integral calculus of complex-valued functions of a complex variable. Here the fundamentals are the Cauchy integral theorem and Cauchy integral formula. These are the basics behind Chapters 2–5.

For Weierstrass, sums and products and especially power series are the central object. These appear first peeking through in the Cauchy chapters (especially Section 2.3) and dominate in Chapters 6, 9, 10, and parts of Chapter 11, Chapter 13, and Chapter 14.

For Riemann, it is the view as conformal maps and associated geometry. The central chapters for this are Chapters 7, 8, and 12, but also parts of Chapters 10 and 11.

In fact, these three strands recur all over and are interrelated, but it is useful to bear in mind the three points of view.

I've made the decision to restrict some results to C^1 or piecewise C^1 curves—for example, we only prove the Jordan curve theorem for that case.

We don't discuss, in this part, boundary values of analytic functions in the unit disk, especially the theory of the Hardy spaces, $H^p(\mathbb{D})$. This is a topic in Part 3. Potential theory has important links to complex analysis, but we've also put it in Part 3 because of the close connection to harmonic functions.

Unlike real analysis, where some basic courses might leave out point set topology or distribution theory, there has been for over 100 years an acknowledged common core of any complex analysis text: the Cauchy integral theorem and its consequences (Chapters 2 and 3), some discussion of harmonic functions on $\mathbb{R}^2$ and of the calculation of indefinite integrals (Chapter 5), some discussion of fractional linear transformations and of conformal maps (Chapters 7 and 8). It is also common to discuss at least Weierstrass product formulas (Chapter 9) and Montel's and/or Vitali's theorems (Chapter 6).

I also feel strongly that global analytic functions belong in a basic course. There are several topics that will be in one or another course, notably the Hadamard product formula (Chapter 9), elliptic functions (Chapter 10), analytic number theory (Chapter 13), and some combination of hypergeometric functions (Chapter 14) and asymptotics (Chapter 15). Nevanlinna theory (Chapter 17) and univalents functions (Chapter 16) are almost always in advanced courses. The break between Parts 2A and 2B is based mainly on what material is covered in Caltech's course, but the material is an integrated whole. I think it unfortunate that asymptotics doesn't seem to have made the cut in courses for pure mathematicians (although the material in Chapters 14 and 15 will be in complex variable courses for applied mathematicians).

Preface to Part 3

I don't have a succinct definition of harmonic analysis or perhaps I have too many. One possibility is that harmonic analysis is what harmonic analysts do. There is an active group of mathematicians, many of them students of or grandstudents of Calderón or Zygmund, who have come to be called harmonic analysts and much of this volume concerns their work or the precursors to that work. One problem with this definition is that, in recent years, this group has branched out to cover certain parts of nonlinear PDE's and combinatorial number theory.

Another approach to a definition is to associate harmonic analysis with "hard analysis," a term introduced by Hardy, who also used "soft analysis" as a pejorative for analysis as the study of abstract infinite-dimensional spaces. There is a dividing line between the use of abstraction, which dominated the analysis of the first half of the twentieth century, and analysis which relies more on inequalities, which regained control in the second half. And there is some truth to the idea that Part 1 in this series of books is more on soft analysis and Part 3 on hard, but, in the end, both parts have many elements of both abstraction and estimates.

Perhaps the best description of this part is that it should really be called "More Real Analysis." With the exception of Chapter 5 on H^p-spaces, any chapter would fit with Part 1—indeed, Chapter 4, which could be called "More Fourier Analysis," started out in Part 1 until I decided to move it here.

The topics that should be in any graduate analysis course and often are, are the results on Hardy–Littlewood maximal functions and the Lebesgue differentiation theorem in Chapter 2, the very basics of harmonic and subharmonic functions, something about H^p-spaces and about Sobolev inequalities.

The other topics are exceedingly useful but are less often in courses, including those at Caltech. Especially in light of Calderón's discovery of its essential equivalence to the Hardy–Littlewood theorem, the maximal ergodic theorem should be taught. And wavelets have earned a place, as well. In any event, there are lots of useful devices to add to our students' toolkits.

Preface to Part 4

The subject of this part is "operator theory." Unlike Parts 1 and 2, where there is general agreement about what we should expect graduate students to know, that is not true of this part.

Putting aside for now Chapters 4 and 6, which go beyond "operator theory" in a narrow sense, one can easily imagine a book titled *Operator Theory* having little overlap with Chapters 2, 3, 5, and 7: almost all of that material studies Hilbert space operators. We do discuss in Chapter 2 the analytic functional calculus on general Banach spaces, and parts of our study of compact operators in Chapter 3 cover some basics and the Riesz–Schauder theory on general Banach spaces. We cover Fredholm operators and the Ringrose structure theory in normed spaces. But the thrust is definitely toward Hilbert space.

Moreover, a book like *Harmonic Analysis of Operators on Hilbert Space*[1] or any of several books with "non-self-adjoint" in their titles have little overlap with this volume. So from our point of view, a more accurate title for this part might be *Operator Theory—Mainly Self-Adjoint and/or Compact Operators on a Hilbert Space.*

That said, much of the material concerning those other topics, undoubtedly important, doesn't belong in "what every mathematician should at least be exposed to in analysis." But, I believe the spectral theorem, at least for bounded operators, the notions of trace and determinant on a Hilbert space, and the basics of the Gel'fand theory of commutative Banach spaces do belong on that list.

[1] See B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, second edition, revised and enlarged edition, Universitext, Springer, New York, 2010.

Before saying a little more about the detailed contents, I should mention that many books with a similar thrust to this book have the name *Functional Analysis*. I still find it remarkable and a little strange that the parts of a graduate analysis course that deal with operator theory are often given this name (since functions are more central to real and complex analysis), but they are, even by me[2].

One change from the other parts in this series of five books is that in them all the material called "Preliminaries" is either from other parts of the series or from prior courses that the student is assumed to have had (e.g., linear algebra or the theory of metric spaces). Here, Chapter 1 includes a section on perturbation theory for eigenvalues of finite matrices because it fits in with a review of linear algebra, not because we imagine many readers are familiar with it.

Chapters 4 and 6 are here as material that I believe all students should see while learning analysis (at least the initial sections), but they are connected to, though rather distinct from, "operator theory." Chapter 4 deals with a subject dear to my heart—orthogonal polynomials—it's officially here because the formal proof we give of the spectral theorem reduces it to the result for Jacobi matrices which we treat by approximation theory for orthogonal polynomials (it should be emphasized that this is only one of seven proofs we sketch). I arranged this, in part, because I felt any first-year graduate student should know the way to derive these from recurrence relations for orthogonal polynomials on the real line. We fill out the chapter with bonus sections on some fascinating aspects of the theory.

Chapter 6 involves another subject that should be on the required list of any mathematician, the Gel'fand theory of commutative Banach algebras. Again, there is a connection to the spectral theorem, justifying the chapter being placed here, but the in-depth look at applications of this theory, while undoubtedly a part of a comprehensive look at analysis, doesn't fit very well under the rubric of operator theory.

[2]See *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press, New York, 1972.

3.5. Classical Fourier Series

I turn away with fear and horror from this lamentable plague of continuous functions that do not have a derivative.

—*Charles Hermite (1822-1901)*
in a letter to Thomas Stieltjes, 1893

Fourier series involve expanding functions periodic with period L in terms of $\{\sin(\frac{2\pi kx}{L})\}_{k=1}^{\infty}$ and $\{\cos(\frac{2\pi kx}{L})\}_{k=0}^{\infty}$. Without loss, we can take $L = 2\pi$ and so consider functions on $\partial\mathbb{D} = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$. The modern approach uses $e^{\pm 2\pi ikx/L}$ rather than sin and cos. The essence of analysis in classical Fourier series is thus ($L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$ is for now defined as in Example 3.1.9 by completion)

Theorem 3.5.1. *$\{e^{ik\theta}\}_{k=-\infty}^{\infty}$ is an orthonormal basis for $L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$.*

Accepting this for a moment, we have, by Theorem 3.4.1, that

Theorem 3.5.2. *For f a continuous function on $\partial\mathbb{D}$, define*

$$f_k^\sharp = \int_0^{2\pi} e^{-ik\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} \tag{3.5.1}$$

Then

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} f_k^\sharp e^{ik\theta} \tag{3.5.2}$$

in the sense that

$$\lim_{K\to\infty} \int_0^{2\pi} \left| f(e^{i\theta}) - \sum_{k=-K}^{K} f_k^\sharp e^{ik\theta} \right|^2 \frac{d\theta}{2\pi} = 0 \tag{3.5.3}$$

and

$$\sum_{k=-\infty}^{\infty} |f_k^\sharp|^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} \tag{3.5.4}$$

Remark. Since $\partial\mathbb{D}$ is compact, f is bounded so all the integrals converge.

This result is sometimes called the Riesz–Fischer theorem. It is not hard to extend this to piecewise continuous functions (Problem 1) and then for a proper choice of f to prove a celebrated formula of Euler (see the Notes) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{3.5.5}$$

(Problem 21).

We will prove Theorem 3.5.1 as a corollary of

Theorem 3.5.3 (Weierstrass Trigonometric Density Theorem). $\{\sum_{k=-K}^{K} a_k e^{ik\theta} \mid \{a_k\}_{k=-K}^{K} \in \mathbb{C}^{2K+1}, K \in \mathbb{N}\}$ *is* $\|\cdot\|_\infty$*-dense in* $C(\partial\mathbb{D})$.

Proof that Theorem 3.5.3 ⇒ Theorem 3.5.1. It is easy to see that if $\varphi_k(e^{i\theta}) = e^{ik\theta}$, then $\langle \varphi_k, \varphi_\ell \rangle = \delta_{k\ell}$. If $\varphi \in L^2$ obeys $\langle \varphi_k, \varphi \rangle = 0$ for all k and $f \in C(X)$ is given, find $\sum_{k=-K}^{K} a_k^{(K)} e^{ik\theta}$ converging in $\|\cdot\|_\infty$ to f. A posteriori, it converges in L^2, so $\langle f, \varphi \rangle = 0$. By construction of L^2, $C(\partial\mathbb{D})$ is dense, so $\langle \varphi, \varphi \rangle = 0$, that is, $\{\varphi_k\}_{k=-\infty}^{\infty}$ is a maximal orthonormal set. □

Theorem 3.5.3 is a restatement of the second density theorem of Weierstrass (Theorem 2.4.2). We'll first prove it using the Stone–Weierstrass theorem. Then we'll find more concrete proofs involving convergence of the Fourier series. We'll give two proofs in the text. In the Problems (see Problems 10, 12, and 3; see also Theorem 3.5.18), we'll provide other results on convergence of Fourier series.

Proof of Theorem 3.5.3 using Stone–Weierstrass. Let $\mathcal{A}$ be the set of finite series of the form $\sum_{k=-K}^{K} a_k e^{ik\theta}$. Since $e^{ik\theta} e^{i\ell\theta} = e^{i(k+\ell)\theta}$, $\mathcal{A}$ is an algebra. Since $\overline{e^{ik\theta}} = e^{-ik\theta}$, $\mathcal{A}$ is closed under conjugation. Since $e^{i\theta}$ separates points on $\partial\mathbb{D}$ and $e^{ik\theta}\big|_{k=0} = 1$, $\mathcal{A}$ obeys all the hypotheses of the complex Stone–Weierstrass theorem (see Theorem 2.5.7), so $\mathcal{A}$ is $\|\cdot\|_\infty$-dense in $C(\partial\mathbb{D})$. □

In the remainder of this section, we'll study three aspects of Fourier series: pointwise or uniform convergence, and so alternate proofs of Theorem 3.5.1; the use of Fourier series to construct nowhere differentiable function; and convergence near discontinuities (an overshoot known as the Gibbs phenomenon).

Given a continuous function, f, on $\partial\mathbb{D}$, define $f_k^\sharp$ by (3.5.1) and the *partial sums* and *Cesàro averages* by

$$S_N(f)(e^{i\theta}) = \sum_{k=-N}^{N} f_k^\sharp e^{ik\theta} \tag{3.5.6}$$

$$C_N(f)(e^{i\theta}) = \frac{1}{N} \sum_{n=0}^{N-1} S_n(f)(e^{i\theta}) \tag{3.5.7}$$

We will prove the following three results about convergence of Fourier series.

Theorem 3.5.4 (Dini's Test). *Let f be a continuous function on $\partial\mathbb{D}$ and let θ_0 be such that*

$$\int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\theta_0})|}{|\theta - \theta_0|} \frac{d\theta}{2\pi} < \infty \tag{3.5.8}$$

Then

$$\lim_{N\to\infty} S_N(f)(e^{i\theta_0}) = f(e^{i\theta_0}) \tag{3.5.9}$$

Remark. See Problem 5 for versions that allow jump discontinuities and don't require f to be continuous away from $e^{i\theta_0}$.

Definition. Let (X,ρ) be a metric space. $f\colon X\to V$, a normed linear space, is called *Hölder continous* of order $\alpha\in(0,1]$ if and only if for some $C>0$ and all $x,y\in X$ with $\rho(x,y)<1$, we have that

$$\|f(x)-f(y)\| \le C\rho(x,y)^\alpha \tag{3.5.10}$$

If $\alpha=1$, f is called *Lipschitz continuous.*

For example, if X is a compact manifold and f is real-valued and differentiable, f is Lipschitz continuous. If (3.5.10) holds for a fixed y and all x with $\rho(x,y)<1$, we say that f is Hölder (or Lipschitz) continuous at y.

Theorem 3.5.5. *Suppose f on $\partial\mathbb{D}$ is complex-valued and Hölder continuous of some order $\alpha>0$. Then $S_N(f)\to f$ uniformly in $C(\partial\mathbb{D})$.*

Remark. The proof shows that it suffices that the *modulus of continuity*, $\Delta_f(\theta)$, defined by

$$\Delta_f(\theta) = \sup_{\substack{e^{i\eta},e^{i\psi}\in\partial\mathbb{D}\\ |\eta-\psi|\le\theta}} |f(e^{i\eta})-f(e^{i\psi})| \tag{3.5.11}$$

obeys a *Dini-type condition*,

$$\int_0^1 \frac{\Delta_f(\theta)\,d\theta}{\theta} < \infty \tag{3.5.12}$$

Hölder continuity says $|\Delta_f(\theta)| \le C|\theta|^\alpha$ obeys (3.5.12), but so does the weaker condition $|\Delta_f(\theta)|\le(\log(\theta^{-1}))^{-\beta}$ for any $\beta>1$.

Since $C^\infty(\partial\mathbb{D})$ is $\|\cdot\|_\infty$-dense in $C(\partial\mathbb{D})$ (see Problem 6) and any C^1 function is Hölder continuous, this proves Theorem 3.5.3, and so Theorem 3.5.1. The following also proves Theorems 3.5.3, and so Theorem 3.5.1. It is not true that $S_N(f)$ converges uniformly to f for all $f\in C(\partial\mathbb{D})$. Indeed, there exist $f\in C(\partial\mathbb{D})$ with $\sup_N\|S_Nf\|_\infty=\infty$ (see Problem 10 of Section 5.4). But for C_N, the situation is different.

Theorem 3.5.6 (Fejér's Theorem)**.** *For any $f\in C(\partial\mathbb{D})$, $C_N(f)\to f$ uniformly.*

We now turn to the proofs of these three theorems. For the first two, we need an "explicit" formula for $S_N(f)$.

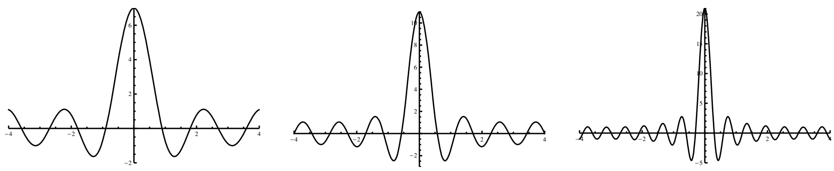

Figure 3.5.1. The Dirichlet kernel for N = 3, 5, 10.

Theorem 3.5.7 (Dirichlet Kernel)**.** *For any continuous function, f, we have*

$$S_N(f)(e^{i\theta}) = \int_0^{2\pi} D_N(\theta-\psi)f(e^{i\psi})\,\frac{d\psi}{2\pi} \tag{3.5.13}$$

where

$$D_N(\eta) = \frac{\sin\bigl[(2N+1)(\frac{\eta}{2})\bigr]}{\sin(\frac{\eta}{2})} \tag{3.5.14}$$

Remarks. 1. Once we have defined L^2 and L^1, (3.5.13) holds for any $f \in L^1$.

2. D_N is called the *Dirichlet kernel.*

3. $D_N(\eta)$ must be invariant under $\eta \to \eta + 2\pi$. While the numerator and denominator of (3.5.14) change sign under this change, the ratio is invariant!

4. See Figure 3.5.1 for plots of D_N for $N = 3, 5, 10$ with scaled y-axis.

Proof. By interchanging the finite sum and integral defining $f_k^\sharp$, we get (3.5.13) where

$$D_N(\eta) = \sum_{k=-N}^{N} e^{ik\eta} \tag{3.5.15}$$

$$= \frac{e^{i(N+1)\eta} - e^{-iN\eta}}{e^{i\eta}-1} \tag{3.5.16}$$

$$= \frac{e^{i(N+\frac{1}{2})\eta} - e^{-i(N+\frac{1}{2})\eta}}{e^{i\eta/2} - e^{-i\eta/2}} \tag{3.5.17}$$

$$= \frac{\sin\bigl[(2N+1)(\frac{\eta}{2})\bigr]}{\sin(\frac{\eta}{2})} \tag{3.5.18}$$

To get (3.5.16), we summed a geometric series, and to get (3.5.17), we multiplied the numerator and denominator by $e^{-i\eta/2}$. □

Proof of Theorem 3.5.4. By rotation covariance, we can suppose, for notational simplicity, that $\theta_0 = 0$, that is, $e^{i\theta_0} = 1$. So using

$$\int_{-\pi}^{\pi} D_N(\theta)\,\frac{d\theta}{2\pi} = \sum_{k=-N}^{N} \int_{-\pi}^{\pi} e^{ik\theta}\,\frac{d\theta}{2\pi} = 1 \tag{3.5.19}$$

we have (using $D_N(0-\theta) = D_N(\theta)$) for all small δ that

$$S_N(f)(1) - f(1) = \int_{-\pi}^{\pi} D_N(\theta)[f(e^{i\theta}) - f(1)]\,\frac{d\theta}{2\pi} \tag{3.5.20}$$

$$= a_N^\delta + b_N^\delta \tag{3.5.21}$$

where a_N^δ is the integral from $-\delta$ to δ and b_N^δ the integral from $-\pi$ to $-\delta$ and δ to π. Since we are focusing on $\theta_0 = 0$, it is convenient to take integrals from $-\pi$ to π rather than 0 to 2π.

Let $g^\delta(e^{i\theta})$ be given by

$$g^\delta(e^{i\theta}) = \begin{cases} 0, & |\theta| < \delta \\ \frac{f(e^{i\theta})-f(1)}{\sin(\frac{\theta}{2})}, & \delta \le |\theta| \le \pi \end{cases} \tag{3.5.22}$$

Let $g_\pm^\delta(e^{i\theta}) = e^{\pm i\theta/2} g^\delta(e^{i\theta})$, so

$$b_N^\delta = \frac{(g_+^\delta)^\sharp_{-N} - (g_-^\delta)^\sharp_N}{2i} \tag{3.5.23}$$

Since, for δ fixed, $g_\pm^\delta$ are bounded, they are in L^2, so $\sum_N |(g_\pm^\delta)^\sharp_N|^2 < \infty$ by (3.5.4). Thus, $\lim_{N\to\infty} (g_\pm^\delta)^\sharp_{\mp N} = 0$. So, for each fixed δ, $b_N^\delta \to 0$ and

$$\limsup_{N\to\infty} |S_N(f)(1) - f(1)| \le \sup_N |a_N^\delta| \tag{3.5.24}$$

$$\le \int_{-\delta}^{\delta} \frac{|f(e^{i\theta}) - f(1)|}{|\sin(\frac{\theta}{2})|}\,\frac{d\theta}{2\pi} \tag{3.5.25}$$

since $|D_N(\theta)| \le |\sin(\frac{\theta}{2})|^{-1}$.

By hypothesis, the integral over all θ is finite, so $\lim_{\delta\downarrow 0}$(RHS of (3.5.25)) $= 0$. Since the left side is δ-independent, we conclude that $\lim_{N\to\infty} |S_N(f)(1) - f(1)| = 0$. □

Proof of Theorem 3.5.5. We sketch the proof, leaving the details to the reader (Problem 8). One looks at the proof above of Theorem 3.5.4 and restores the θ_0-dependence. Since we have a bound on $\sup_{|\theta-\psi|\le\delta} |f(e^{i\theta}) - f(e^{i\psi})| \equiv \Delta_f(\delta)$ that obeys (3.5.12), the $a_N^\delta(\theta_0), b_N^\delta(\theta_0)$ terms obey $\sup_{\theta_0,N} |a_N^\delta(\theta_0)| \to 0$ as $\delta \downarrow 0$, so one only needs, for each fixed $\delta > 0$, that

$$\lim_{N\to\infty} \big(\sup_{\theta_0} |b_N^\delta(\theta_0)|\big) = 0 \tag{3.5.26}$$

One first shows that if $\{h_\alpha(e^{i\theta})\}$ is a compact set of h_α's in L^2, then $(h_\alpha^\sharp)_n \to 0$ uniformly in α, and then that $g^\delta_{\pm,\theta_0}$ is continuous in $e^{i\theta_0} \in \mathbb{D}$ to get compactness (see Problem 7). $\square$

The argument that $b_N^\delta \to 0$ in the proof of Theorem 3.5.4 implies a nice localization result going back to Riemann:

Theorem 3.5.8 (Riemann Localization Principle)**.** *Assume that f is in $L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$ and for some θ_0 and some $\varepsilon > 0$, $f(e^{i\theta}) = 0$ for $|\theta - \theta_0| < \varepsilon$. Then $(S_N f)(e^{i\theta_0}) \to 0$ as $N \to \infty$. In particular, if f and g are in L^2 and equal near $e^{i\theta_0}$ and $(S_N f)(e^{i\theta_0})$ has a limit, then $(S_N g)(e^{i\theta})$ has the same limit.*

Remark. Once we have L^1 and the more general Riemann–Lebesgue lemma (Theorem 6.5.3), this extends to L^1.

Finally, to prove Fejér's theorem, we need an analog of (3.5.9) for the Cesàro averages, $C_N(f)$:

Theorem 3.5.9 (Fejér Kernel)**.** *For any continuous function, f, we have*

$$C_N(f)(e^{i\theta}) = \int_0^{2\pi} F_N(\theta - \psi) f(e^{i\psi}) \, \frac{d\psi}{2\pi} \tag{3.5.27}$$

where

$$F_N(\eta) = \frac{1}{N} \left[\frac{\sin(\frac{N\eta}{2})}{\sin(\frac{\eta}{2})} \right]^2 \tag{3.5.28}$$

Remark. See Figure 3.5.2 for plots of C_N for $N = 3, 5, 10$.

Proof. By Theorem 3.5.7, we have (3.5.27) where

$$F_N(\eta) = \frac{1}{N} \sum_{j=0}^{N-1} D_j(\eta) \tag{3.5.29}$$

$$= \frac{1}{N \sin(\frac{\eta}{2})} \, \mathrm{Im}\left(\sum_{j=0}^{N-1} e^{i(j+\frac{1}{2})\eta} \right) \tag{3.5.30}$$

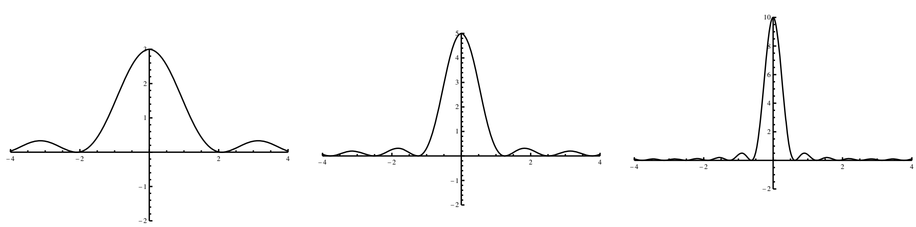

Figure 3.5.2. The Fejér kernel for N = 3, 5, 10.

$$= \frac{1}{N\sin(\frac{\eta}{2})}\ \mathrm{Im}\left[\frac{e^{i(N+\frac{1}{2})\eta} - e^{i\eta/2}}{e^{i\eta}-1}\right] \tag{3.5.31}$$

$$= \frac{1}{N\sin(\frac{\eta}{2})}\ \mathrm{Im}\left[\frac{e^{iN\eta} - 1}{e^{i\eta/2}-e^{-i\eta/2}}\right] \tag{3.5.32}$$

$$= \frac{(-1)}{N\sin^2(\frac{\eta}{2})}\ \frac{1}{2}\ \mathrm{Re}[e^{iN\eta}-1] \tag{3.5.33}$$

$$= \frac{(-1)}{N\sin^2(\frac{\eta}{2})}\ \frac{1}{2}\ \mathrm{Re}\left[2i\sin\left(\frac{N\eta}{2}\right)e^{iN\eta/2}\right] \tag{3.5.34}$$

$$= \frac{1}{N\sin^2(\frac{\eta}{2})}\ \sin^2\left(\frac{N\eta}{2}\right) \tag{3.5.35}$$

$$= \text{RHS of (3.5.28)} \tag{3.5.36}$$

We get (3.5.31) by summing a geometric series, (3.5.32) by multiplying numerator and denominator by $e^{-i\eta/2}$, (3.5.33) from $e^{i\eta/2} - e^{-i\eta/2} = 2i\sin(\frac{\eta}{2})$, (3.5.34) by $(x^2-1) = (x-x^{-1})x$, and (3.5.35) by $\mathrm{Re}[ie^{ia}] = -\sin(a)$. □

Proposition 3.5.10. *$g_N(\eta) \equiv F_N(\eta)$ obeys*

(i)

$$g_N(\eta) \geq 0 \tag{3.5.37}$$

(ii)

$$\int_0^{2\pi} g_N(\eta)\,\frac{d\eta}{2\pi} = 1 \tag{3.5.38}$$

(iii) *For any $\varepsilon > 0$,*

$$\lim_{N\to\infty}\int_{\varepsilon<\eta<2\pi-\varepsilon} g_N(\eta)\,\frac{d\eta}{2\pi} = 0 \tag{3.5.39}$$

Proof. (i) is trivial and (ii) is immediate from (3.5.19) and (3.5.29). Since $|\sin(\frac{\eta N}{2})| \leq 1$ and $\sin^2(\frac{\eta}{2})$ is monotone increasing on $[0,\pi]$ and decreasing on $[\pi, 2\pi]$, we have

$$F_N(\eta) \leq \frac{1}{N\sin^2(\frac{\varepsilon}{2})} \qquad \text{if } \varepsilon < \eta < 2\pi - \varepsilon \tag{3.5.40}$$

from which (iii) is immediate. □

Definition. A sequence of continuous functions $\{g_N\}_{N=1}^\infty$ on $\partial\mathbb{D}$ obeying (i)–(iii) of Proposition 3.5.10 is called an *approximate identity.*

Theorem 3.5.11. *If $\{g_N\}_{N=1}^\infty$ is an approximate identity and $f \in C(\partial\mathbb{D})$, then*

$$g_N * f \to f \tag{3.5.41}$$

uniformly on $\partial\mathbb{D}$, where

$$(h * f)(e^{i\theta}) = \int_0^{2\pi} h(\theta - \psi) f(e^{i\psi}) \frac{d\psi}{2\pi} \tag{3.5.42}$$

Remarks. 1. One application of this is to prove that $C^\infty(\partial\mathbb{D})$ is $\|\cdot\|_\infty$-dense in $C(\partial\mathbb{D})$; see Problem 6.

2. This result is only stated for continuous $\{g_N\}_{N=1}^\infty$ because, at this point, we only know how to integrate continuous functions. Once one has L^1, one can define L^1 approximate identities by the above definition with "continuous" replaced by L^1. This theorem extends with no change in the proof.

Proof. By periodicity and (3.5.38),

$$(g_N * f)(e^{i\theta}) - f(e^{i\theta}) = \int_0^{2\pi} g_N(\psi)[f(e^{i(\theta-\psi)}) - f(e^{i\theta})] \frac{d\psi}{2\pi} \tag{3.5.43}$$

so breaking the integral into $0 < \psi < \varepsilon$ or $2\pi - \varepsilon < \psi < 2\pi$ and its complement, we get

$$\|g_N * f - f\|_\infty \le \sup_{|\psi|<\varepsilon} |f(e^{i(\theta-\psi)}) - f(e^{i\theta})| + 2\|f\|_\infty \int_\varepsilon^{2\pi-\varepsilon} g_N(\psi) \frac{d\psi}{2\pi} \tag{3.5.44}$$

using (i) and (ii) of the definition of approximate identity.

By property (iii),

$$\limsup_{N\to\infty} \|g_N * f - f\|_\infty \le \sup_{|\psi|<\varepsilon} |f(e^{i(\theta-\psi)}) - f(e^{i\theta})| \tag{3.5.45}$$

Since f is continuous, it is uniformly continuous (by Theorem 2.3.10), so the sup goes to zero as $\varepsilon \downarrow 0$. □

Proof of Theorem 3.5.6. Immediate from Theorems 3.5.9 and 3.5.11 and Proposition 3.5.10, □

There is nothing special about $\partial\mathbb{D}$.

Definition. A sequence of functions, $\{g_N(x)\}_{N=1}^\infty$ on $\mathbb{R}^\nu$ is called an *approximate identity* if and only if

(i) $g_N(x) \ge 0$ (3.5.46)

(ii) $\displaystyle\int g_N(x)\, d^\nu x = 1$ (3.5.47)

(iii) For any $\varepsilon > 0$,

$$\lim_{N\to\infty} \int_{|x|\ge\varepsilon} g_N(x)\, d^\nu x = 0 \tag{3.5.48}$$

If h and g are functions on $\mathbb{R}^\nu$, one defines their convolution by

$$(h * g)(x) = \int h(y)g(x-y)\, d^\nu y \tag{3.5.49}$$

$$= \int h(x-y)g(y)\, d^\nu y \tag{3.5.50}$$

Note that if $\int g(x)\, d^\nu x < \infty$ and $\|h\|_\infty < \infty$, then the integrals converge uniformly and absolutely.

The same argument that led to Theorem 3.5.11 implies

Theorem 3.5.12. *Let $\{g_N\}_{N=1}^\infty$ be an approximate identity. If f is bounded and uniformly continuous on $\mathbb{R}^\nu$, then as $N \to \infty$,*

$$g_N * f \overset{\|\cdot\|_\infty}{\longrightarrow} f \tag{3.5.51}$$

If there is a compact set $K \subset \mathbb{R}^\nu$ so $\operatorname{supp}(g_N) \subset K$ for all N and f is continuous (but not necessarily bounded or uniformly continuous on all of $\mathbb{R}^\nu$), then

$$\lim_{N\to\infty} (g_N * f)(x) = f(x) \tag{3.5.52}$$

uniformly for x in each compact subset of $\mathbb{R}^\nu$.

With the Fejér kernel in hand, we can construct examples of nowhere differentiable continuous functions of the form first studied by Weierstrass. We'll consider

$$f(x) = \sum_{n=1}^\infty a^n n^\gamma \cos(b^n x) \tag{3.5.53}$$

where b is an integer with $b \geq 2$, $\gamma \in \mathbb{R}$, and $0 < a < 1$ or $a = 1$, $\gamma < -1$. Since $|a| < 1$ (or $a = 1$, $\gamma < -1$), the finite sum converges uniformly, and so f is a continuous periodic function. We'll prove below that if $ab > 1$, f is nowhere differentiable. The key is that the Fourier coefficients have large gaps, so we not only have that $\frac{1}{2}a^n n^\gamma = \int e^{-ib^n x} f(x)\, dx$, but we can insert $F_N(x)$ if $N < b^n - b^{n-1}$ in front of f without changing the integral. The key will then be:

Lemma 3.5.13. *There exists constant c_α, $0 < \alpha \leq 1$, so that for all $N \geq 2$,*

$$(2\pi)^{-1} \int_{-\pi}^{\pi} |x|^\alpha F_N(x)\, dx \leq \begin{cases} c_\alpha N^{-\alpha} & 0 < \alpha < 1 \\ c_1 \frac{\log(N)}{N} & \alpha = 1 \end{cases} \tag{3.5.54}$$

Proof. Clearly, by (3.5.15), $|D_N(x)| \leq D_N(0)$, so

$$|F_N(x)| \leq F_N(0) = N \tag{3.5.55}$$

Since $\lim_{\eta\to 0}$, $|\eta|/|\sin\eta| = 1$ and, by computing derivatives, increasing on $(0,\pi/2)$, $|\sin\eta| \geq (2/\pi)|\eta|$ for $|\eta| \leq \pi/2$. It follows that

$$|F_N(x)| \leq \frac{\pi^2}{Nx^2} \tag{3.5.56}$$

(3.5.54) follows by using (3.5.55) on $\{x \mid |x| \leq 1/N\}$ and (3.5.56) on $\{x \mid 1/N \leq |x| \leq \pi\}$. □

Proposition 3.5.14. *Let $f(x)$ be an L^2 function on $(-\pi,\pi)$ with Fourier coefficients $f_j^\sharp$. Suppose that f is extended periodically to $\mathbb{R}$, and for some x_0, $C > 0$ and $\alpha \in (0,1]$, we have that for all x,*

$$|f(x) - f(x_0)| \leq C|x - x_0|^\alpha \tag{3.5.57}$$

Suppose also that for some $k \neq 0$ and N with $1 < N < |k|$, we have that

$$f_j^\sharp = 0 \qquad \text{for } 0 < |j - k| \leq N - 1 \tag{3.5.58}$$

Then, with c_α given by (3.5.54),

$$|f_k^\sharp| \leq \begin{cases} Cc_\alpha N^{-\alpha}, & 0 < \alpha < 1 \\ Cc_1 \frac{\log(N)}{N}, & \alpha = 1 \end{cases} \tag{3.5.59}$$

Proof. Since shifting x by $-x_0$ multiplies $f_j^\sharp$ by a phase factor, we can suppose $x_0 = 0$. Since replacing f by $f - f(0)$ doesn't change $f_j^\sharp$ for $j \neq 0$ or (3.5.57), we can suppose $x_0 = 0$ and $f(x_0) = 0$.

Let $\varphi_j(x) = e^{ijx}$. Then since $F_N(x)$ is a linear combination of $\{\varphi_\ell\}_{\ell=-(N-1)}^{N-1}$ with constant term 1,

$$\varphi_k F_N = \varphi_k + \text{linear combination of } \{\varphi_\ell\}_{1\leq|\ell-k|\leq N-1}$$

so

$$f_k^\sharp = \langle \varphi_k, f\rangle = \langle \varphi_k F_N, f\rangle \tag{3.5.60}$$

so that

$$|f_k^\sharp| \leq (2\pi)^{-1} \int |F_N(x)|\,|f(x)|\,dx \tag{3.5.61}$$

$$\leq C(2\pi)^{-1} \int |F_N(x)|\,|x|^\alpha\,dx \tag{3.5.62}$$

so (3.5.54) implies (3.5.59). □

Write the function in (3.5.53) as $f_{a,b,\gamma}$. Then

Theorem 3.5.15. (a) *For $0 < a \leq 1$, $f_{a,b,\gamma}$ is Hölder continuous of order α if $ab^\alpha < 1$ or $ab^\alpha = 1$, $\gamma < -1$. If $\alpha = 1$ and these conditions hold, $f_{a,b,\gamma}$ is C^1.*

(b) *For $0 < \alpha < 1$, if $ab^\alpha > 1$ or $ab^\alpha = 1$ and $\gamma > 0$, then $f_{a,b,\gamma}$ is nowhere Hölder continuous of order α. If $ab > 1$ or $ab = 1$ and $\gamma > 1$, then f is nowhere Lipschitz and, in particular, nowhere differentiable.*

Proof. (a) Since $|\cos x - \cos y| \le 2$ and

$$|\cos x - \cos y| \le \left| \int_x^y \sin u \, du \right| \le |x - y|$$

we have for any $\alpha \in [0,1]$ that $|\cos x - \cos y| \le 2^{1-\alpha}|x-y|^\alpha$, so

$$|f_{a,b,\gamma}(x) - f_{a,b,\gamma}(y)| \le 2^{1-\alpha}|x-y|^\alpha \sum_{n=1}^\infty a^n b^{n\alpha} n^\gamma \tag{3.5.63}$$

If either $ab^\alpha < 1$ or $ab^\alpha = 1$ and $\gamma < -1$, the sum in (3.5.63) converges and we get global Hölder continuity.

(b) We consider the case $\alpha = 1$. $\alpha < 1$ is similar (Problem 16). If $f_{a,b,\gamma}$ is Lipschitz continuous at some x_0, by Proposition 3.5.14 and (3.5.59) with $k = b^n$ and $N = b^n - b^{n-1} = b^n(1 - b^{-1})$, we get that for some constant K,

$$a^n n^\gamma \le K n b^{-n} \tag{3.5.64}$$

or

$$(ab)^n n^{\gamma - 1} \le K \tag{3.5.65}$$

If $ab > 1$ or $ab = 1$ and $\gamma > 1$, this is false for n large, so $f_{a,b,\gamma}$ cannot be Lipschitz at any point. □

Example 3.5.16. For $a = \frac{1}{2}$, $b = 2$, and $\gamma = 2$, $f_{a,b,\gamma}$ is nowhere Lipschitz continuous (and so nowhere differentiable) but Hölder continuous for all $\alpha < 1$. This result is true also for $\gamma = 0$; see the Notes and Problem 17.

Fix α_0 with $0 < \alpha_0 < 1$. For $a = (\frac{1}{2})^{\alpha_0}$, $b = 2$, $\gamma = 2$, $f_{a,b,\gamma}$ is Hölder continuous for $\alpha < \alpha_0$ and nowhere Hölder continuous for $\alpha \ge \alpha_0$. If instead, $\gamma = -2$, one gets Hölder continuity for $\alpha \le \alpha_0$ and nowhere Hölder continuous for $\alpha > \alpha_0$.

If $b = 2$, $a = 1$, $\gamma = -2$, $f_{a,b,\gamma}$ is continuous, but nowhere Hölder continuous for any $\alpha > 0$. □

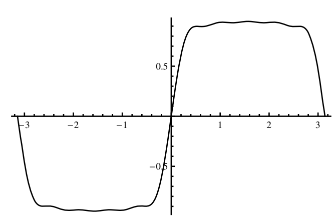

Figure 3.5.3. C_{11} for a step function.

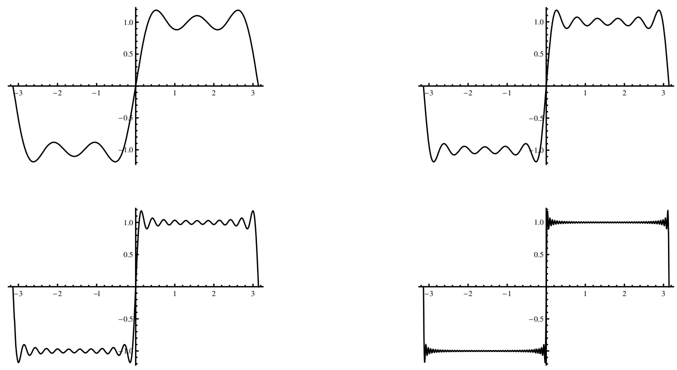

Figure 3.5.4. S_n, $n = 5, 12, 21, 101$ for a step function.

Finally, we turn to an aspect of convergence of Fourier series known as the *Gibbs phenomenon*. Consider the function on $\partial\mathbb{D}$,

$$f(e^{i\theta}) = \begin{cases} 1, & 0 < \theta < \pi \\ -1, & \pi < \theta < 2\pi \\ 0, & \theta = 0 \text{ or } \pi \end{cases} \tag{3.5.66}$$

By Dini's test extended to nonglobally continuous functions (Problem 9), $S_n(f)(e^{i\theta}) \to f(e^{i\theta})$ uniformly on each set $\{e^{i\theta} \mid \varepsilon < \theta \le \pi - \varepsilon, \pi + \varepsilon < \theta < 2\pi - \varepsilon\}$ for $\varepsilon > 0$. Since $f(e^{-i\theta}) = -f(e^{i\theta})$, $f^\sharp_{-k} = -\overline{f^\sharp_k}$, so $S_n(f)(1) = S_n(f)(-1) = 0$. Thus, one might think that $S_n(f)$ for n large looks like the graph in Figure 3.5.3, hugging f closely except for a linear piece extending from -1 to 1 or 1 to -1 at $\theta = 0, \pi$. Indeed, this is what happens for $C_n(f)$—in fact, Figure 3.5.3 is $C_{11}(f)$ and the reader will prove $\|C_n(f)\|_\infty \le \|f\|_\infty$ in Problem 20. However, Figure 3.5.4 plots $S_n(f)$ for $n = 5, 12, 21, 101$. The Gibbs phenomenon is the systematic overshoots shown in this figure.

Theorem 3.5.17 (Gibbs Phenomenon)**.** *For the step function, f, given by (3.5.66),*

$$\lim_{n\to\infty} \|S_n f\|_\infty = \frac{2}{\pi} \int_0^\pi \frac{\sin s}{s}\, ds = 1.178979744\ldots \tag{3.5.67}$$

Moreover, the points where $|S_n f|$ is maximal are given by $\pm(\pi/n + O(1/n^2))$.

Sketch. We'll leave the justifications to the Problems (Problem 21). By a simple calculation, we have

$$f^\sharp_{2n} = 0, \qquad f^\sharp_{2n+1} = \frac{2}{i(2n+1)\pi} \tag{3.5.68}$$

so using the fact that

$$\frac{1}{i(2j+1)}\, e^{(2j+1)ix} = \int_0^x e^{(2j+1)it}\, dt + \frac{1}{i(2j+1)} \tag{3.5.69}$$

and the cancellation of the constants, one finds that, after summing a geometric series,

$$(S_{2n}f)(x) = (S_{2n-1}f)(x) = \frac{2}{\pi}\int_0^x \frac{\sin(2nt)}{\sin t}\, dt \tag{3.5.70}$$

$$= G(2nx) + O(x^2) \tag{3.5.71}$$

where $O(x^2)$ means an error bounded by Cx^2 uniformly in n, and where

$$G(y) = \frac{2}{\pi}\int_0^y \frac{\sin s}{s}\, ds$$

This comes from $1/\sin t - 1/t = O(t)$.

Since $S_n(f) \to f$ uniformly away from $0, \pi$, we see that if $\pm y_\infty$ are the points where $|G(y)|$ is maximum, then $\lim_{n\to\infty}\|S_n f\|_\infty$ is $\sup|G(y)|$ and the maximum point is $\pm y_\infty/2n + O(1/n^2)$.

Since $G'(y) = (2\sin y)/\pi y$, the relative maxima of $|G|$ occur at multiples of π, and using the oscillations and decay of y^{-1}, one sees that the maximum occurs at $y_\infty = \pi$ with $\sup|G(y)| = G(\pi)$. □

Notes and Historical Remarks.

> Apart from his prefectorial duties Fourier helped organise the "Description of Egypt" ... Fourier's main contribution was the general introduction—a survey of Egyptian history up to modern times. (An Egyptologist with whom I discussed this described the introduction as a masterpiece and a turning point in the subject. He was surprised to hear that Fourier also had a reputation as a mathematician.)
>
> —*T. W. Körner* [**512**]

We are hampered in this section by the fact that we only discuss measurable and L^p functions in the next chapter. So we've made use of the vague term "function" without descriptive adjectives. For now, we interpret this as continuous functions. But we emphasize, as the reader should check after L^1 is defined, that Theorems 3.5.11 and 3.5.12 are valid if the g_N's are only L^1 functions (with all the formal properties of an approximate identity).

Fourier series are such a fundamental part of analysis that there are many books devoted solely or at least substantially to them. Among these are [**263**, **309**, **356**, **357**, **479**, **512**, **871**, **875**, **935**, **974**, **1024**]. In particular, Zgymund [**1024**] remains a readable classic.

The history of Fourier analysis is intimately wrapped up with an understanding of what a function is, and later, which functions have integrals. In the early history, a key role was played by Euler and the Bernoullis. Part 2A has capsule biographies for them (Section 9.2 for Euler and Section 9.7 for the Bernoulli family).

The early history revolved around the wave equation in one dimension, $\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$. (We use units in which the wave speed is 1; the eighteenth-century work had a speed of propagation.) In about 1750, d'Alembert [**212**] and Euler [**285**] independently found general solutions of the form $f(x-t) + g(x+t)$, where f and g are "arbitrary functions." The eighteenth-century notion of function meant given by an explicit analytic expression involving sums, powers, trigonometric functions, and the like. A sharp controversy partially in letters and partially in papers developed. Euler argued that you needed to allow an initial condition like $u(x,0) = \frac{1}{2} - |\frac{1}{2} - \frac{x}{\pi}|$ on $[0,\pi]$, thinking of a plucked string which was viewed as two analytic expressions, ($\frac{x}{\pi}$ on $(0,\frac{\pi}{2})$ and $1-\frac{x}{\pi}$ on $(\frac{\pi}{2},\pi)$), and d'Alembert didn't like that.

Shortly after that, Daniel Bernoulli [**78**], following a 1715 observation of Brook Taylor [**913**], pointed out that $\cos(kt)\sin(kx)$ is also a solution (shifting variables for our $(0,\pi)$ case), and if one wanted $u(\pm\pi, z) = 0$ boundary conditions, one could take $k = 1, 2, \dots$. He claimed that the d'Alembert–Euler solutions could be represented as sums of solutions of this $\cos(nt)\sin(nx)$ form. There followed lively exchanges among the three, joined also by Lagrange and then Laplace, that involved what kind of functions could be represented by infinite sums of sines and cosines. Euler argued that only functions with a single expression could be so represented—which was ironic given that he had elsewhere considered the sums that converge to the jump, as we do in Theorem 3.5.17. Only Bernoulli was in the "any function can" camp. This issue of what kinds of functions Fourier sums could represent stayed open until the work of Dirichlet (and, even more broadly, of Riesz–Fischer) discussed below. Because of its importance to the understanding of functions, and to the history of Fourier analysis and of waves in physics, this controversy has seen considerable historical analysis: see Ravetz [**760**], Grattan-Guinness [**362**], and Wheeler–Crummett [**986**].

In his work on planetary motion, Euler [**286**] also used sine and cosine sums. Using orthogonality and formal interchange of sum and integral, he essentially found the formula (3.5.1) for the coefficients (he used $\sin(k\theta)$ and $\cos(k\theta)$, not $e^{\pm ik\theta}$).

In 1799, Parseval [**700**] also considered such sums and wrote what was essentially (3.5.4) without any explicit proof or calculations. So, on the basis of this work, one of only five published works, Marc-Antoine Parseval des Chênes (1755–1836) is known to posterity. For example, we used his name for the abstract Hilbert space result, (3.4.3). We also used the name of Michel Plancherel (1885–1967), a Swiss mathematician, who in 1910 [**730**] provided one of the first proofs of the analog for Fourier transforms and thereby got his name on all sorts of L^2 relations of transforms, such as (3.4.3).

Next in the picture was Jean Baptiste Joseph Fourier (1768–1830). Fourier was more a physicist than a mathematician and his engineering expertise led to high political appointments. He started life as the ninth child of a tailor and became a baron of the First French Empire. He was active in revolutionary politics and was imprisoned during the reign of terror. It is likely that it was only the fall of Robespierre that prevented him from losing his head long before his scientific discoveries! He was involved with Napoleon's 1798–99 campaign in Egypt, starting as scientific adviser and ending as governor of Lower Egypt. In 1801, Napoleon appointed him as prefect (administrative head) of a province that included Grenoble, where he lived, supervising the construction of a highway from Grenoble to Turin, among other tasks. He initially supported the new king at the time of Napoleon's escape from Elba and had to flee Grenoble to avoid Napoleon's army. He then shifted back to Napoleon and was distrusted by the king after Waterloo, enough so that for a time, the king prevented his election to the French Academy. After Waterloo, he returned to Paris, and in 1822 he became the secretary of the Academy. For more on his life, see Körner [**512**, Sects. 92–93] and Herival [**419**].

Undoubtedly, Fourier is most known for his book on heat [**311**] written in 1804–07, while he was prefect in Grenoble. He submitted it to the French Academy in 1807. He used what we now call Fourier series and the Fourier transform (see Sections 6.3 and 6.5) in solving the heat equation (see Section 6.9). His claims about expanding arbitrary functions were only one of the controversial elements of his book, leading the committee of Lagrange, Laplace, Monge, and Lacroix to hold up publication. Along the way, the work got a prize from a committee of Lagrange, Laplace, Malus, Haüy, and Legendre. It was finally published in 1822.

This book established the usefulness of the method and many basic formulae. One of Fourier's results was the sin/cos version of (3.5.1), which he found not knowing of Euler's earlier derivation. Unlike Euler, who used orthogonality, Fourier's proof was very complicated and involved expanding sine in a Taylor series, collecting terms, and manipulating the power series for f—a procedure especially questionable for the discontinuous functions Fourier claimed one could expand in Fourier series!

The validity of Fourier expansions was established by the seminal paper [**249**] of Johann Peter Gustav Lejeune Dirichlet (1805–59). A capsule biography of Dirichlet appears in the Notes to Section 13.4 of Part 2B. We note here that this paper was published in 1829 when Dirichlet was only twenty-four years old, that he studied under Fourier in Paris, and that Fourier was instrumental in Dirichlet getting a position in Germany around that time.

Dirichlet used his kernel to show that many noncontinuous functions, f, had convergent Fourier series, with the requirement that the limit at the point of discontinuity is $\frac{1}{2}(f(x+0)+f(x-0))$ (see Problem 5). He supposed his functions were continuous except at finitely many points, smooth in between (exactly how smooth wasn't made explicit), had left and right limits at the points of discontinuity, and had only finitely many maxima and minima. We now know these conditions are overkill—smoothness by itself is enough, as is the maximum-minimum condition alone if interpreted as functions of bounded variation (see below). Nevertheless, Dirichlet's result was radical for its time. Shortly before, in one his texts on Analysis, Cauchy had claimed that a pointwise limit of continuous functions is continuous. It took the clarifying notion of uniform convergence (pushed by Weierstrass) to settle these questions.

We note that in 1873, Paul du Bois-Reymond (1831–1889) [**256**] constructed a continuous function on $\partial\mathbb{D}$ whose Fourier series was divergent at a given point. Fejér [**297**] found a different example of this sort and in Problem 4 we expose his idea. (In Problem 10 of Section 5.4, the reader will show there exists $f \in C(\partial\mathbb{D})$ so $\|S_N f\|_\infty \to \infty$, a closely related fact. In Problem 12 of that section, the reader will prove that, in the language of that section, a Baire generic function has $|(S_N f)(1)| \to \infty$ and in Problem 13 that for a Baire generic function, $|(S_N f)(e^{i\theta})| \to \infty$ for a Baire generic set of θ.)

Dirichlet's work set the baseline for all later work on Fourier series convergence, of which we want to mention five: that of Dini, Jordan, Fejér, Riesz–Fischer, and Carleson.

Ulisse Dini (1845–1918) wrote a book on Fourier series [**247**] that includes Theorem 3.5.4. (3.5.8) is called the *Dini test* or *Dini condition.* Occasionally, a function that obeys (3.5.12) is called *Dini continuous*.

Another basic convergence theorem is due to Camille Jordan (1838–1922) [**455**]:

Theorem 3.5.18 (Jordan's Theorem)**.** *If f is a function of bounded variation on $\partial\mathbb{D}$, then $S_n(f(e^{i\theta})) \to \frac{1}{2}\big[f(e^{i(\theta+0)}) + f(e^{i(\theta-0)})\big]$ for any $x \in (0,1)$.*

Functions of bounded variation (which were first defined in this paper of Jordan) are defined and discussed in Sections 4.1 and 4.15. In particular, Theorem 4.15.2 shows any such function is a difference of monotone functions, so it is sufficient to prove Jordan's theorem for monotone functions, which the reader does in Problem 3.

Lipót Fejér (1880–1959) proved Theorem 3.5.6 along the lines we do in his 1900 paper [**298**], written when he was only nineteen. For a discussion of the impact of his discovery on the revival of interest in Fourier analysis, see Kahane [**465**]. Fejér was born Lipót Weiss (German for "white") and was a student of Hermann Schwarz (German for "black"). He changed his name to Fejér (archaic Hungarian for "white") around 1900 and one of his students was Fekete (Hungarian for "black"). Among Fejér's other students were Paul Erdős, George Pólya, Tibor Radö, Marcel Riesz, Gabor Szegő, Paul Turán, and John von Neumann. Fejér spent most of his career at the University of Budapest, although he initially had trouble with his appointment because he was Jewish. He suffered during the Nazi occupation of Hungary in 1944, treatment that it is believed led to a loss of his mental capacity after the Second World War.

The last of the classical convergence results is the fact we regard as the definition of Fourier expansion, namely, for any $f \in L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$, $\int |(S_n f)(e^{i\theta}) - f(e^{i\theta})|^2 \frac{d\theta}{2\pi} \to 0$, a result sometimes called the Riesz–Fischer theorem after [**775**, **305**]. These papers completed the story of which functions can be represented as Fourier series. To do this, the authors needed to prove completeness of L^2 (defined as classes of measurable functions), and it is this that we (along with many others) will call the Riesz–Fischer theorem. We discuss it further in Section 4.4 and its Notes.

In 1928, M. Riesz proved that for $1 < p < \infty$, for $f \in L^p(\partial\mathbb{D}, \frac{d\theta}{2\pi})$, we have $\|f - S_n f\|_p \to 0$ [**790**]. We'll prove this in Section 5.8 of Part 3. For $p = 1$ or ∞, it is known that there are f's in L^p with $\|S_N f\|_p \to \infty$; see Problem 10 of Section 5.4.

No discussion of pointwise convergence would be complete without mention of Lennart Carleson's (1928–) famous 1966 result [**162**] that for any f in $L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$, $(S_N f)(e^{i\theta})$ converges to f for Lebesgue a.e. θ. This result is a famous conjecture of Lusin and was extended to all L^p, $p > 1$, by Hunt [**438**]. As mentioned, for a generic continuous function $\big(S_N f\big)(e^{i\theta})$ diverges on a (dense) generic set, but, by Carleson's theorem, one of Lebesgue measure zero.

For $p = 1$, Kolmogorov [**501**] gave an $L^1(\partial\mathbb{D}, \frac{d\theta}{2\pi})$ function whose Fourier series diverges at every point in $\partial\mathbb{D}$. Three years earlier, when he was twenty-one, Kolmogorov found a similar function with almost everywhere divergence. Katznelson [**479**, Sect. II.3.5] has a proof of this result using the de la Vallée Poussin kernel of Problem 14. While the proof of Carleson's theorem is beyond the scope of these volumes, we'll prove a related result in Section 2.11 of Part 3: namely, if f has $f^\sharp \in \ell^p$, $1 \le p < 2$, then for a.e. θ, $(S_N f)(e^{i\theta}) \to f(e^{i\theta})$ ($p = 2$ is Carleson's theorem).

In Problem 12, an alternate result to Fejér's theorem is presented, proving abelian limits of Fourier series to a continuous function. It is due to Picard and Fatou (see the remark to the problem). Littlewood [**596**] has proven that if $\sum_{n=0}^{N} a_n$ has an abelian limit α and $|a_n| \le C(n+1)^{-1}$, then the sum itself converges to α (see Section 6.11 of Part 4, especially Problem 5). Thus, any continuous function, f, on $\partial\mathbb{D}$ with $f_n^\sharp = O(n^{-1})$ has a convergent Fourier series. By an integration by parts in a Stieltjes integral, it is easy see if f has bounded variation $f_n^\sharp = O(n^{-1})$, so this provides another proof of Jordan's theorem.

Underlying Fourier series is a group structure. $\partial\mathbb{D}$ is a group under multiplication $e^{i\theta_1}, e^{i\theta_2} \mapsto e^{i(\theta_1+\theta_2)}$ and $d\omega/2\pi$ is the unique measure invariant under this multiplication. The functions $\varphi_n(e^{i\theta}) = e^{in\theta}$ are exactly the only continuous functions, χ, on $\partial\mathbb{D}$ obeying

$$\chi(e^{iy}e^{ix}) = \chi(e^{iy})\chi(e^{ix}) \tag{3.5.72}$$

Extensions of Fourier series where the group is $\mathbb{R}^\nu$ will occur in Chapter 6 while general locally compact abelian groups will appear in Section 6.9 of Part 4.

Relevant to this section is the group, $\mathbb{Z}_N$, a cyclic group of order N thought of as $\mathbb{Z}/N\mathbb{Z}$, for integers mod N. Given f on $\mathbb{Z}$ of period N, we define

$$(\mathcal{F}_N f)(m) = \frac{1}{N}\sum_{j=0}^{N-1} f(j)\bar{\omega}_N^{mj} \tag{3.5.73}$$

where ω_N is a primitive Nth root of unity, i.e.,

$$\omega_N = \exp(2\pi i/N) \tag{3.5.74}$$

Since $\varphi_j(m) = \omega^{mj}$ are an orthonormal basis for functions on $\{1,\dots,N\}$ (with $\langle c,d\rangle = \frac{1}{N}\sum_{i=1}^N \bar{c}_i d_i$ inner product), the inverse is

$$(\mathcal{F}_N^{-1} h)(m) = \sum_{j=0}^{N-1} h(j)\omega_N^{mj} \tag{3.5.75}$$

$\mathcal{F}_N$ is called the *discrete Fourier transform.*

Clearly, if f is continuous on $\partial\mathbb{D}$ and $f_N(j) = f(\omega_N^j)$, then $\mathcal{F}_N f_N \to f^\sharp$ pointwise, so $\mathcal{F}_N$ is of interest not only for its own sake but as a method of numerical approximation of the map $f \mapsto f^\sharp$. In this regard, there is an important algorithm for $\mathcal{F}_N$ called the *Fast Fourier Transform* (FFT).

The purpose of the FFT is to dramatically reduce the number of computations to get $\mathcal{F}_N$ from $O(N^2)$ to $O(N\log N)$ at least when $N = 2^m$ (so for $m = 20$, i.e., $N \approx 1{,}000{,}000$) from about a trillion calculations to more like twenty million! Since multiplication is much slower than addition, we'll only count multiplications and we'll ignore the N multiplications needed to get the powers $\{\omega_N^j\}_{j=0}^{N-1}$ given ω_N.

If one uses (3.5.73) naively, one needs N^2 multiplications (of $\bar{\omega}_N^{mj}$ and $f(j)$). If one writes

$$(\mathcal{F}_{2N}f)(m) = \frac{1}{2N}\sum_{j=0}^{N-1} f(2j)\bar{\omega}_{2N}^{2mj} + \frac{\bar{\omega}_{2N}^m}{2N}\sum_{j=0}^{N-1} f(2j\pi)\bar{\omega}_{2N}^{2mj} \tag{3.5.76}$$

and defines f_e and f_0 (for sum and add) on $\{0, 1, \dots, N-1\}$ by

$$f_e(j) = f(2j), \quad f_0(j) = f(2j+1) \tag{3.5.77}$$

then

$$(\mathcal{F}_{2N}f)(m) = \frac{1}{2}\left(\mathcal{F}_N f_e\right)(m) + \frac{\bar{\omega}_{2N}^m}{2}\left(\mathcal{F}_N f_0\right)(m) \tag{3.5.78}$$

if $0 \le m < N$ and, if $N \le m \le 2N-1$.

If we have an algorithm to compute $\mathcal{F}_N$ in a_N multiplication steps, we can compute $\mathcal{F}_{2N}$ in

$$a_{2N} = 2a_N + N \tag{3.5.79}$$

multiplication steps (the N comes from the N multiplications by ω_{2N}^m). When $N = 2^{\ell-1}$, we can iterate ℓ times and use $a_1 = 1$ to get

$$a_{2^\ell} = (l+1)2^\ell \tag{3.5.80}$$

yielding to $O(N \log N)$ algorithm.

This algorithm was popularized by and is sometimes named after a 1965 paper of Cooley–Tukey [**204**]. They rediscovered an idea that Gauss knew about—it appeared in Gauss' complete works as an unpublished note. The Cooley–Tukey algorithm came at exactly the right time—just as digital computers became powerful enough to compute Fourier transforms of data important in the real world, and there was an explosion of applications. In fact, Tukey came up with the basic algorithm as a member of President Kennedy's Presidential Scientific Advisory Committee to try to figure out a way to analyze seismic data in order to get information on Russian nuclear tests! Garwin from IBM, also at the meeting, put Tukey in touch with Cooley who actually coded the algorithm!

One reason that Weierstrass' example had such impact is that earlier in the century, Ampère [**26**] seemed to claim that every continuous function was differentiable. Medvedev [**647**, Ch. 5] in a summary of these developments, argues that the problem was one of terminology. When Ampère wrote, neither "function" nor "continuous" had clearly accepted definitions and, Medvedev says, Ampère had in mind functions given locally by convergent power series! Shortly afterwards, Cauchy gave more careful notions (and Weierstrass, later, even more so). Be that as it may, many mid-century analysis texts stated and proved (!) what they called Ampère's theorem: that every continuous function was differentiable. In his lectures as early as the 1860s, Weierstrass claimed that all these proofs were wrong.

The first results on the existence of nondifferentiable continuous functions are due to Bernhard Bolzano (1781–1848), a Czech priest (his father was from Italy). He found them around 1830 but never published them—they were finally published about a hundred years later; see Pinkus [**728**] for details. Around 1880, Charles Cellérier (1818–89) proved that for a large positive integer, a, the function $f(x) = \sum_{n=1}^{\infty} a^{-n}\sin(a^n x)$ is continuous but nowhere differentiable. He never published the result but it was discovered among his papers and published posthumously [**178**].

Weierstrass [**978**] claimed that in lectures given in 1861, Riemann asserted that $\sum_{n=1}^{\infty} n^{-2}\sin(n^2 x)$, a function that enters in elliptic function theory, was continuous but nondifferentiable on a dense set. It is now known to be nondifferentiable, except for an explicit countable set. Weierstrass couldn't verify Riemann's claim. Instead, in 1872, he considered the function in (3.5.53) for $\gamma = 0$ and proved that if $a < 1$, b is an odd integer, and if $ab > 1 + \frac{3}{2}\pi$, then f is nowhere differentiable. This example of Weierstrass had a profound effect on his contemporaries.

There were intermediate improvements by Bromwich, Darboux, Dini, Faber, Hobson, Landsberg, and Lerch, until Hardy [**393**] got the definitive result $ab \geq 1$ (and it is differentiable if $ab < 1$). In the text, we only handled $ab > 1$; $ab = 1$ (and $\gamma = 0$) can be handled using the Jackson kernel related to the square of the Fejér kernel (see Problem 17). I don't know who found this Fejér- and Jackson-kernel approach, but I've found it in several books from the 1960s.

There are close connections between nowhere differentiable functions and natural boundaries, especially lacunary series; see Problem 16 of Section 2.3 of Part 2A and Kahane [**464**].

A sign of the roughness of the functions $f_{a,b,\gamma}$ is that their graphs (i.e., $\{(x,y) \mid y = f(x)\}$) have dimension greater than one. Indeed, it is known that with a suitable definition of dimension ("box dimension," believed also for Hausdorff dimension; see Section 8.2), then for $ab > 1$ and b sufficiently large,

$$\dim(\operatorname{graph}(f_{a,b,\gamma=0})) = 2 - \frac{\log(a^{-1})}{\log(b)}$$

This is discussed in Falconer [**293**]; see Figure 3.5.5. For extensive additional literature on nowhere differentiable functions, see the bibliography at `http://mathworld.wolfram.com/WeierstrassFunction.html`.

The Gibbs phenomenon is named after J. Willard Gibbs (1839–1903), the famous American physicist known for his work on statistical mechanics

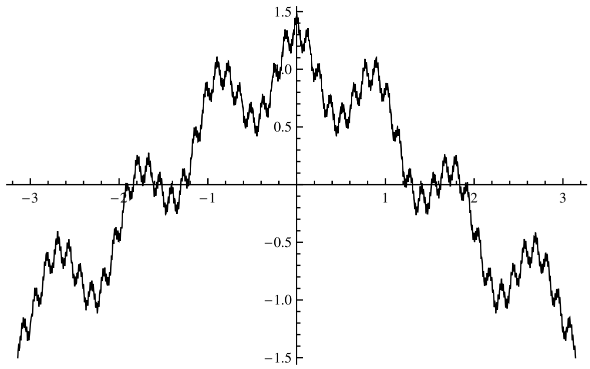

Figure 3.5.5. $F_{a,b,\gamma}$ for $a = \frac{1}{3}$, $b = 7$. This graph has dimension approximately 1.44.

after his paper [**348**]. It was so named by Maxime Bôcher (1867–1918), who found the first comprehensive mathematical treatment [**98**, **99**], much like the one we sketch. The name is a good example of Arnold's principle, since fifty years before Gibbs, Henry Wilbraham (1825–83) discovered the phenomenon [**1001**]; see Hewitt–Hewitt [**423**] for the history. The Gibbs phenomenon has been rediscovered many times, for example, by engineers working on radar during the Second World War.

(3.5.5) was proven in several ways first by Euler in 1734–35 thereby solving a famous problem; see the discussion in the Notes to Sections 5.7 and 9.2 of Part 2A. Even though Euler proved (3.5.4) (much later), he doesn't seem to have noticed the connection.

Problems

1. Let f be piecewise continuous on $\partial\mathbb{D}$ in that there are $0 \le \theta_1 < \cdots < \theta_k < 2\pi$, so $f(e^{i\theta})$ is continuous at any $e^{i\theta_0} \in \partial\mathbb{D} \setminus \{e^{i\theta_j}\}_{j=1}^k$, and for any k, $\lim_{\varepsilon\downarrow 0} f(e^{i(\theta_k+\varepsilon)}) \equiv f(e^{i(\theta_k+0)})$ and $\lim_{\varepsilon\uparrow 0} f(e^{i(\theta_k-\varepsilon)}) \equiv f(e^{i(\theta_k-0)})$ exist.

   (a) Prove there are continuous f_n on $\partial\mathbb{D}$ so $\int_0^{2\pi} |f(e^{i\theta}) - f_n(e^{i\theta})|^2 \frac{d\theta}{2\pi} \to 0$ as $n \to \infty$.

   (b) Prove that if $f_k^\sharp$ is defined by (3.5.1), then (3.5.4) holds.

2. In this problem, the reader will prove that for all N and $0 < a < b < 2\pi$
$$\left| \int_a^b D_N(x)\, dx \right| \le 4\pi \tag{3.5.81}$$
   a result that will be useful in the next two problems.

   (a) Prove it suffices to prove this for $0 < a < b < \pi$ with 4π replaced by 2π. (*Hint*: $D_N(2\pi - x) = D_N(x)$.)

(b) For $0 < x < \pi - \frac{2\pi}{(N+\frac{1}{2})}$, show that $D_N(x)$ and $D_N(x + \frac{2\pi}{(N+\frac{1}{2})})$ have opposite signs with $|D_N(x)| > |D_N(x + \frac{2\pi}{(N+\frac{1}{2})})|$ and use this to prove for $0 < a < b < \pi$, the integral has maximum absolute value for $a = 0$, $b = \pi/(N + \frac{1}{2})$ (Look at the right halves of the graph in Figure 3.5.1).

(c) Prove that $\left| \int_0^{\pi/(N+\frac{1}{2})} D_N(x)\,dx \right| \le D_N(0) \frac{\pi}{(N-\frac{1}{2})} = 2\pi$.

Remark. We'll see later (Problem 10 in Section 5.4) that $\sup_N \int_0^{2\pi} |D_N(\theta)| \frac{d\theta}{2\pi} = \infty$.

3. This problem will prove Theorem 3.5.18. You'll need to know about functions of bounded variation (see Sections 4.1 and 4.15) and the second mean value theorem (see Problem 5 of Section 4.15). Since any function of bounded variation is a difference of monotone increasing functions (see Theorem 4.15.2), you can suppose that $f(e^{i\theta})$ is monotone in θ on $[-\pi, \pi]$.

   (a) For each $x_0 \in [-\pi, \pi]$, show that it suffices to find a small δ so that, as $N \to \infty$,

$$\int_{x_0}^{x_0+\delta} f(e^{ix}) D_N(x_0 - x)\,dx \to \tfrac{1}{2} f(e^{i(x_0+0)})$$
$$\int_{x_0-\delta}^{x_0} f(e^{ix}) D_N(x_0 - x)\,dx \to \tfrac{1}{2} f(e^{i(x_0-0)})$$

   (b) Prove that it suffices to show for g monotone on $[0, \delta]$ and $g(0) = g(0+) = 0$ then, as $N \to \infty$,

$$\int_0^\delta g(x) D_N(x)\;dx \to g(0) = 0 \tag{3.5.82}$$

   (c) For some $c \in (0, \delta)$, prove that

$$\int_0^\delta g(x) D_N(x)\,dx = g(\delta-) \int_c^\delta D_N(x)\,dx$$

   (d) Prove $\limsup |\int_0^\delta g(x) D_N(x)\,dx| \le 4\pi g(\delta-)$. (*Hint*: Use Problem 2.)

   (e) For any $0 < \delta' < \delta$, prove that $\lim_{N\to\infty} |\int_{\delta'}^\delta g(x) D_N(x)\,dx| = 0$. (*Hint*: Look at the proof of Theorem 3.5.8.)

   (f) Prove (3.5.82), and so, Jordan's theorem.

4. This problem will construct (following ideas of Fejér [**297**]) a continuous function f on $\partial\mathbb{D}$ so that $\overline{\lim}\, S_N(f)(0) = \infty$.

(a) As a preliminary, prove that for all n and $x \in [-\pi, \pi]$

$$\left| \sum_{k=1}^{n} \frac{\sin(kx)}{k} \right| \leq \frac{3\pi}{2} \tag{3.5.83}$$

(*Hint*: Show that the sum is $\frac{1}{2}\int_0^x (D_n(t) - 1)\, dt$ and use Problem 2.)

(b) Define

$$G_n(\theta) = \sum_{j=0}^{n-1} \frac{1}{n-j} \left[e^{ij\theta} - e^{i(2n-j)\theta} \right] \tag{3.5.84}$$

Prove that uniformly in n for $\theta \in [-\pi, \pi]$

$$|G_n(\theta)| \leq 3\pi$$

(c) Now pick $0 < n_1 < n_2 < \dots$ and $m_1, m_2, \dots$ so that $m_k > m_{k-1} + 2n_{k-1}$ and a sequence of positive numbers $\{a_k\}_{k=1}^\infty$ with $\sum_{k=1}^\infty a_k < \infty$ and let

$$f(e^{i\theta}) = \sum_{k=1}^{\infty} a_k \, e^{im_k\theta} \, G_{n_k}(\theta) \tag{3.5.85}$$

Show the sum is absolutely and uniformly convergent so that f is a continuous function.

(d) Prove that $\sum_{j=1}^n j^{-1} > \log(n+1)$.

(e) Prove that if $N_k = m_k + n_k$, then

$$\big(S_{N_k} f\big)(\theta = 0) \geq a_k \, \log(n_k + 1) - \sum_{j=1}^{\infty} a_j \tag{3.5.86}$$

(f) Pick $n_k = 2^{k^3} = m_k$ and $a_k = k^{-2}$ and show that $\big(S_{N_k} f\big)(\theta = 0) \to \infty$.

5. This problem supposes you know about elements of L^2 as Borel functions, as discussed in Sections 4.4 and 4.6.

(a) Suppose that $f \in L^2(\partial\mathbb{D})$ and for some θ_0 and δ, we have

$$\int_{|\theta-\theta_0|\leq\delta} \frac{|f(e^{i\theta}) - f(e^{i\theta_0})|}{|\theta - \theta_0|} \frac{d\theta}{2\pi} < \infty \tag{3.5.87}$$

Prove that (3.5.9) holds. (*Hint*: See Theorem 3.5.8.)

(b) Suppose that instead of (3.5.87) you have $f_\pm = \lim_{\varepsilon\downarrow 0} f(e^{i(\theta_0 \pm \varepsilon)})$ exists and

$$\int_{\theta_0}^{\theta_0+\delta} \frac{|f(e^{i\theta}) - f_+|}{|\theta - \theta_0|} \frac{d\theta}{2\pi} + \int_{\theta_0-\delta}^{\theta_0} \frac{|f(e^{i\theta}) - f_-|}{|\theta - \theta_0|} \frac{d\theta}{2\pi} < \infty \tag{3.5.88}$$

Prove that

$$(S_N f)(e^{i\theta_0}) \to \tfrac{1}{2}\,(f_+ + f_-) \tag{3.5.89}$$

(*Hint*: Find g with $g(\theta) = -g(-\theta)$, so $(S_N f)(1) \equiv 0$ and so that $h(e^{i\theta}) \equiv f(e^{i\theta}) - g(\theta - \theta_0)$ is continuous at θ_0 and obeys (3.5.87).)

6. (a) Let h be C^∞ on $\partial\mathbb{D}$ and f continuous. Prove that $h * f$ is C^∞.

   (b) By constructing C^∞ approximate identities, prove $C^\infty(\partial\mathbb{D})$ is $\|\cdot\|_\infty$ dense in $C(\partial\mathbb{D})$.

7. Let K be a compact subset of $L^2(\partial\mathbb{D})$. Prove that for any ε, there is an N so that for all $f \in K$ and $n \geq N$, $|f_n^\sharp| \leq \varepsilon$. (*Hint*: First find $f^{(1)}, \dots, f^{(n)}$ so that $K \subset \cup_{j=1}^{\ell}\{g \mid \|g - f^{(j)}\|_2 \leq \frac{\varepsilon}{2}\}$.)

8. Fill in the details of the proof of Theorem 3.5.5.

9. Suppose for some open interval $I \subset \partial\mathbb{D}$ and $f \in L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$, we have

$$\sup_{\theta\in I} \int \frac{|f(e^{i\psi}) - f(e^{i\theta})|}{|\psi - \theta|} \frac{d\psi}{2\pi} < \infty$$

   Prove that for every compact $K \subset I$, we have $\sup_{\theta\in K}|S_N f(e^{i\theta}) - f(e^{i\theta})| \to 0$.

10. (a) Suppose $\sum_{n\in\mathbb{Z}}|a_n| < \infty$. Prove that $\sum_{|n|\leq N} a_n e^{in\theta} \equiv g_N(\theta)$ converges uniformly to a continuous function $g(\theta)$ on $\partial\mathbb{D}$.

    (b) If f is C^1 on $\partial\mathbb{D}$, prove that $(f')_n^\sharp = inf_n^\sharp$.

    (c) If f is C^1 on $\partial\mathbb{D}$, prove that $\sum_{n\in\mathbb{Z}}(1 + |n|^2)|f_n^\sharp|^2 < \infty$.

    (d) If f is C^1 on $\partial\mathbb{D}$, prove that $\sum_{n\in\mathbb{Z}}|f_n^\sharp| < \infty$.

    (e) If f is C^1 on $\partial\mathbb{D}$, prove that $S_N(f)$ converges uniformly to f. (*Hint*: If g is the uniform limit of $S_N(f)$, prove that $g^\sharp = f^\sharp$, and then that $f = g$.)

    Remark. There exist f's in $C(\partial\mathbb{D})$ for which $\sum_{n\in\mathbb{Z}}|f_n^\sharp| = \infty$; see Problem 10(e) and the Notes to Section 6.7.

11. (a) Prove that $\{1, \cos\theta, \sin\theta\}$ is a Korovkin set in the sense discussed in Theorem 2.4.7. (*Hint*: $|e^{i\theta} - e^{i\theta_0}|^2$.)

    (b) Use Korovkin's theorem to prove Fejér's theorem.

12. This shows that abelian summation, rather than Cesàro summation, provides uniform convergence of Fourier series, and so provides yet another proof of Theorem 3.5.3. Given f a continuous function on $\partial\mathbb{D}$, define the Abel sum of the Fourier series for each $a > 0$ by

$$(A_a f)(e^{i\theta}) = \sum_{n=-\infty}^{\infty} e^{-a|n|} f_n^\sharp e^{in\theta} \tag{3.5.90}$$

(a) Prove that

$$(A_a f)(e^{i\theta}) = \int_0^{2\pi} P_a(\theta - \psi) f(e^{i\psi}) \, \frac{d\psi}{2\pi} \tag{3.5.91}$$

where

$$P_a(\theta) = \frac{1 - e^{-2a}}{1 + e^{-2a} - 2e^{-a} \cos \theta} \tag{3.5.92}$$

known as the *Poisson kernel*.

(b) Prove that $\{P_a(\theta)\}$ is an approximate identity as $a \downarrow 0$ (with an obvious extension of the notion to continuous a rather than discrete n).

(c) Conclude that for any $f \in C(\partial\mathbb{D})$, $A_a f \to f$ uniformly as $a \downarrow 0$.

Remark. This proof of the second Weierstrass theorem is due to Picard [**723**]. In one of the first papers applying Lebesgue's theory, this approach was extended by Fatou [**295**] in his 1906 thesis. It had earlier been used by Lebesgue himself in proving uniqueness of Fourier coefficients. We will have a lot more to say about the Poisson kernel in Section 5.3 of Part 2A and Sections 2.4 and 3.1, and Chapter 5 of Part 3. Part 3 will discuss an analog of this problem for spherical harmonic expansions.

13. This provides another proof of Theorem 3.5.3. The approximate identity is simpler than Fejér's, although without the direct Fourier series interpretation.

(a) Let

$$\gamma_n = \int_{-\pi}^{\pi} (1 + \cos \theta)^n \, \frac{d\theta}{2\pi}$$

Prove that $W_n(\theta) = \gamma_n^{-1}(1 + \cos\theta)^n$ is an approximate identity.

(b) For any continuous f, prove that $f * W_n$ is of the form $\sum_{j=-n}^{n} a_j^{(n)} e^{ij\theta}$. Conclude that Theorem 3.5.3 holds.

Remarks. 1. This proof of the second Weierstrass theorem is due to de la Vallée Poussin [**229**].

2. This is sometimes written as $W_n(\theta) = \tilde{\gamma}_n^{-1} \cos^{2n}(\frac{\theta}{2})$.

14. This will prove that given any $f \in L^1(\partial\mathbb{D}, \frac{d\theta}{2\pi})$, there is a sequence of trigonometric polynomials (i.e, finite sums of $e^{ij\theta}$, $j \in \mathbb{Z}$), $P_n(e^{i\theta})$, so that (i) $\|P_n\|_1 \le 3\|f\|_1$; (ii) $P_n^\sharp(k) = f^\sharp(k)$ if $|k| \le n$; (iii) $f^\sharp(k) = 0$ if $|k| \ge 2n$.

(a) Define the de la Vallée Poussin kernel, $V_n(\eta)$, by

$$V_n(\eta) = 2F_{2n-1}(\eta) - F_{n-1}(\eta) \tag{3.5.93}$$

where F_n is the Fejér kernel. Prove that

$$F_n^\sharp(j) = \begin{cases} 1 - \frac{|j|}{n+1} & \text{if } |j| \le n \\ 0 & \text{if } |j| \ge n+1 \end{cases} \tag{3.5.94}$$

and

$$V_n^\sharp(j) = \begin{cases} 1 & \text{if } |j| \le n \\ 2 - \frac{|j|}{n} & \text{if } n+1 \le |j| \le 2n-1 \\ 0 & \text{if } |j| \ge 2n \end{cases} \tag{3.5.95}$$

(b) Prove $\|V_n\|_{L^1} \le 3$ for all n.

(c) If $P_n = V_n * f$, prove that P_n has the properties (i)–(iii).

Remark. The de la Vallée Poussin kernel first appeared in his 1918 paper [**230**].

15. This will lead the reader through a proof of the classical Weierstrass approximation theorem (see Section 2.4) due to Landau [**542**]. The *Landau kernel* is defined by

$$L_n(x) = \begin{cases} \gamma_n^{-1}(1-x^2)^n, & |x| \le 1 \\ 0, & |x| \ge 1 \end{cases} \tag{3.5.96}$$

where

$$\gamma_n = \int_{-1}^{1} (1-x^2)^n \, dx \tag{3.5.97}$$

(a) Prove that $2 \ge \gamma_n \ge Cn^{-1}$ for some C. (*Hint*: $1 - x^2 \ge (1 - |x|)$ and use $y = x/n$; *Remark*: In fact (see Theorem 15.2.2 of Part 2B), $\gamma_n \sim Cn^{-1/2}$.)

(b) Prove that L_n is an approximate identity for $\mathbb{R}$, so that if f is a continuous function on $\mathbb{R}$ with compact support, then $f * L_n \to f$ uniformly.

(c) Let

$$\tilde{L}_n(x) = \gamma_n^{-1}(1-x^2)^n \qquad \text{for all } x \tag{3.5.98}$$

For f continuous with $\text{supp}(f) \subset [-\frac{1}{2}, \frac{1}{2}]$, prove that

$$\int f(y)[L_n(x-y) - \tilde{L}_n(x-y)] \, dy = 0 \tag{3.5.99}$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$.

(d) Conclude for such f that $\tilde{L}_n * f \to f$ uniformly on $[-\frac{1}{2}, \frac{1}{2}]$. Prove that $\tilde{L}_n * f$ is a polynomial in x.

(e) If f is a continuous function on $[-\frac{1}{2}, \frac{1}{2}]$, prove that there are α, β so $f(x) - \alpha x - \beta$ vanishes at $\pm\frac{1}{2}$, and conclude that f is a uniform limit on $[-\frac{1}{2}, \frac{1}{2}]$ of polynomials in x.

(f) Prove the Weierstrass theorem for any interval.

16. Prove Theorem 3.5.15(b) when $0 < \alpha < 1$.

17. The *Jackson kernel* is defined by

$$J_N(\theta) = \gamma_N^{-1} F_N(\theta)^2 \tag{3.5.100}$$

where

$$\gamma_N = \int F_N(\theta)^2 \, \frac{d\theta}{2\pi} \tag{3.5.101}$$

(a) Prove that $(J_N)_k^\sharp = 1$ if $k = 0$ and $= 0$ if $k > 2(N-1)$.

(b) Prove that if f obeys

$$f_j^\sharp = 0 \quad \text{for } 0 < |j - k| < 2(N-1) \tag{3.5.102}$$

then

$$|f_k^\sharp| \le (2\pi)^{-1} \int J_N(x) |f(x)| \, dx \tag{3.5.103}$$

(c) Prove that $\gamma_N > N/2$. (*Hint*: Look at the Fourier coefficients of F_N.)

(d) For some constant, c_1, prove that

$$|J_N(x)| \le \frac{c_1}{N^3 x^4} \tag{3.5.104}$$

(e) For some constant, c_2, prove that

$$\int_{-\pi}^{\pi} J_N(x) |f(x)| \, \frac{d\theta}{2\pi} \le c_2 \biggl(N^{-2} \int |f(x)| \, dx + N^{-1} \sup_{|x| \le N^{-1/4}} |f(x)| \biggr) \tag{3.5.105}$$

(*Hint*: For $|x| \le N^{-1}$, use $\int_{-\pi}^{\pi} J_N(x) \, \frac{dx}{2\pi} = 1$; for $N^{-1} \le |x| \le N^{-1/4}$, use (3.5.104) and $\int_{N^{-1}}^{N^{-1/4}} t^{-3} \, dt \le N^2$; for $N^{-1/4} \le |x| \le \pi$, use (3.5.104) to see $\sup_{|x| \ge N^{-1/4}} |J_N(x)| \le c_2 N^{-2}$.)

(f) If f is continuous and Lipschitz at some point and obeys (3.5.102), prove for some constant, c_3, that

$$|f_k^\sharp| \le c_3 (N^{-2} + o(N^{-1})) \tag{3.5.106}$$

(g) Prove that if $ab = 1$, $a < 1$, then $f_{a,b,\theta=0}$ is nowhere differentiable.

Remark. The Jackson kernel is named after Dunham Jackson (1888–1946), an American mathematician who spent most of his career at the University of Minnesota. He introduced his kernel in his 1912 dissertation done under Edmund Landau. The index on it is sometimes one-half the one used in this problem, so that in (3.5.102), twice the index is replaced by the index.

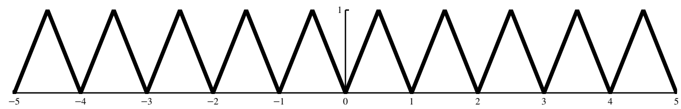

Figure 3.5.6. A tent function.

18. This problem will construct what is probably the simplest nowhere differentiable function (or perhaps the variant in the next problem). For $x \in \mathbb{R}$, let $Q(x) = 2\,\mathrm{dist}(x, \mathbb{Z})$, a period 1 "tent function" (see Figure 3.5.6). Let

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} Q(2^n x) \tag{3.5.107}$$

(a) Suppose g is any function differentiable at some point x and $y_n \le x \le z_n$, where $y_n \neq z_n$ and $\lim_{n\to\infty}(z_n - y_n) = 0$. Prove that

$$\lim_{n\to\infty} \frac{g(z_n) - g(y_n)}{z_n - y_n} \to g'(x)$$

(b) Prove that f, given by the sum in (3.5.107), defines a continuous function on $\mathbb{R}$.

(c) Let $\mathbb{D}_\ell = \{j/2^\ell \mid j \in \mathbb{Z}\}$ be the dyadic rationals of order ℓ, and define for any $x \in \mathbb{R}$, $y_\ell(x), z_\ell(x) \in \mathbb{D}_\ell$ by $y_\ell(x) = 2^{-\ell}[2^\ell x]$ and $z_\ell(x) = y_\ell(x) + 1/2^\ell$. Prove that $y_\ell(x) \le x \le z_\ell(x)$.

(d) For any x, prove that if $m \ge \ell$, then $Q(2^m y_\ell(x)) = Q(2^m z_\ell(x)) = 0$.

(e) Let $\widetilde{R}(x) = 2\chi_{[0,\frac{1}{2})}(x) - 2\chi_{[\frac{1}{2},1)}(x)$ and $R(x) = \sum_{n\in\mathbb{Z}} \widetilde{R}(x-n)$. For any $m < \ell$ and any $x \in \mathbb{R}$, prove that $2^{-m}[Q(2^m z_\ell(x)) - Q(2^m y_\ell(x))] = 2^{-\ell} R(2^m x)$. (*Hint*: If $Q_m(x) = 2^{-n} Q(2^m x)$, prove $Q_m(y) - Q_m(z) = \int_y^z Q_m'(w)\,dw$, where Q_m' exists for all but a discrete set of points, and then that on $[y_\ell(x), z_\ell(x)]$, we have (except for a discrete set) that $Q_m'(x) = R(2^m x)$. Note that you'll need to give careful consideration to the case where $x \in \mathbb{D}_\ell$.)

(f) Let $q_n(x) = [f(z_n(x)) - f(y_n(x))]/[z_n(x) - y_n(x)]$. Prove that $q_n(x) = \sum_{j=0}^{n-1} R(2^n x)$ and conclude that $|q_{n+1}(x) - q_n(x)| = 2$ for all x and n. Show that f is nowhere differentiable.

Remark. This function is due to Takagi [**903**] in 1903, although the example is sometimes named after van der Waerden who rediscovered it (with 2^n replaced by 10^n) twenty-five years later. The function is sometimes called the *blancmange function* since its graph looks like the French dessert of that name (see Figure 3.5.7). This approach is from de Rham [**237**]. It is known that $\{x \mid f(x) = \sup_y f(y)\}$ is an uncountable set of Hausdorff dimension $\frac{1}{2}$; see Baba [**43**].

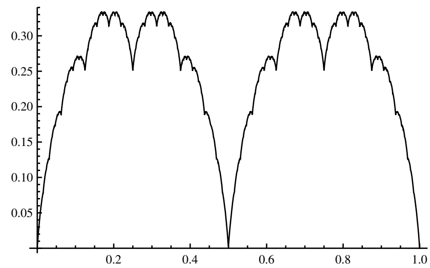

Figure 3.5.7. The Takagi function.

19. This has a variant of the Takagi function of Problem 18 due to McCarthy [**645**]. Let $g_n(x) = 2Q(\frac{1}{4}2^{2^n}x)$, where Q is the tent function of Problem 18. Let

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} g_n(x) \qquad (3.5.108)$$

(a) Prove that f is continuous.

(b) Prove that g_k has constant slope $\pm 2^{2^k}$ on intervals of size $2\,2^{-2^k}$ and has period $4\,2^{-2^k}$.

(c) Given k and x, pick $\Delta_k x = \pm 2^{-2^k}$ so x and $x + \Delta_k x$ lie in a single interval where g_k has constant slope. Prove this can be done and that, if $(\Delta_k h) = h(x + \Delta_k x) - h(x)$, then $|\Delta_k g_k| = 1$.

(d) For $n > k$, prove that $\Delta_k g_n = 0$. (*Hint*: The period of g_n divides $\Delta_k x$.)

(e) For $n < k$, prove that $|\Delta_k g_n| \leq 2^{-2^{(k-1)}}$. (*Hint*: Look at g_n'.)

(f) Prove that

$$\frac{\sum_{n \neq k} 2^{-n} |\Delta_k g_n|}{|2^{-k} \Delta_k g_k|} \leq 2^{k+1} 2^{-2^{(k-1)}}$$

(g) Prove that $\Delta f / 2^{-k} \Delta_k g_k \to 1$ as $k \to \infty$.

(h) Prove that $|\Delta f| / |\Delta_k x| \to \infty$ as $k \to \infty$ and conclude that f is nowhere differentiable.

Remark. McCarthy seems to have been unaware of the work of Takagi and van der Waerden and, in turn, de Rham seems to have been unaware of McCarthy.

20. Prove that $\|C_N f\|_\infty \leq \|f\|_\infty$.

21. This problem will fill in the details of the proof of Theorem 3.5.17 and also prove (3.5.5).

(a) If f is given by (3.5.66), verify (3.5.68) for $n = 0, \pm 1, \pm 2, \dots$.

(b) Prove (3.5.70).

(c) Prove (3.5.71).

(d) Complete the proof of Theorem 3.5.17.

(e) Prove that $\sum_{n=0}^\infty \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$. (*Hint*: Use (3.5.68) and (3.5.4).)

(f) If $S = \sum_{n=1}^\infty \frac{1}{n^2}$ and $E = \sum_{n=0}^\infty \frac{1}{(2n+1)^2}$, prove that $S = E + \frac{1}{4}S$.

(g) Prove (3.5.5).

22. (a) Compute $g_n^\sharp$ if $g(\theta) = |\theta - \frac{\pi}{2}|$ on $[0, 2\pi]$.

(b) Verify that $\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$.

23. (a) Let h be given on $[0, 2\pi]$ by

$$h(\theta) = \begin{cases} \theta(\pi - \theta), & 0 \le \theta \le \pi \\ (\pi - \theta)(2\pi - \theta), & \pi \le \theta \le 2\pi \end{cases}$$

Compute $h_n^\sharp$.

(b) Verify that $\sum_{n=1}^\infty \frac{1}{n^6} = \frac{\pi^6}{945}$.

Remark. Problem 4 of Section 9.2 of Part 2A will find $\sum_{n=1}^\infty \frac{1}{n^{2k}}$ for all k (in terms of rationals known as the Bernoulli numbers).

24. Suppose that for some C, $\alpha > 0$, and $\varepsilon > 0$, we have $0 < x < y < \varepsilon$ or $0 > x > y > -\varepsilon \Rightarrow |f(x) - f(y)| \le C|x - y|^\alpha$, and that $\lim_{\delta \downarrow 0} f(\pm\delta) \equiv f(\pm 0)$ exist. Let $\Delta = |f(+0) - f(-0)|$. Prove that

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \Big[\sup_{|x| \le \delta} (S_n f)(x) - \inf_{|x| \le \delta} S_n(f) \Big] = \bigg(\frac{2}{\pi} \int_0^\pi \frac{\sin s}{s}\, ds \bigg) \Delta$$

showing that the Gibbs phenomenon is generally true at jumps.

25. This problem will prove *Wirtinger's inequality*: if $f(e^{i\theta})$ is a C^1 real-valued function on $\partial\mathbb{D}$ with

$$f(1) = f(-1) = 0 \tag{3.5.109}$$

then

$$\int_0^{2\pi} |f(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \le \int_0^{2\pi} |f'(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \tag{3.5.110}$$

You'll also prove (3.5.110) if

$$\int_0^{2\pi} f(e^{i\theta}) \, \frac{d\theta}{2\pi} = 0 \tag{3.5.111}$$

(a) Compute $(f')_n^\sharp$ in terms of $f_n^\sharp$ and deduce (3.5.110) if (3.5.111) holds.

(b) Suppose next that

$$f(e^{i\theta}) = -f(e^{-i\theta}) \tag{3.5.112}$$

Prove that (3.5.110) holds.

(c) Given any f obeying (3.5.109), find C^1 g, h obeying (3.5.112) so $f \restriction \{e^{i\theta} \mid 0 \le \theta \le \pi\} = g \restriction \{e^{i\theta} \mid 0 \le \theta \le \pi\}$, $f \restriction \{e^{i\theta} \mid -\pi \le \theta \le 0\} = h \restriction \{e^{i\theta} \mid -\pi \le \theta \le 0\}$. Using (3.5.110) for g and h, prove it for f.

(d) Prove that when (3.5.109) holds, equality holds in (3.5.110) only if $f(e^{i\theta}) = \sin\theta$.

Remark. (3.5.110) was noted by Wirtinger if either (3.5.109) or (3.5.111) holds, but he never published it. He mentioned it to Blaschke who included it in his famous book on geometric inequalities [**95**].

26. This problem will prove a version of the *isoperimetric inequality*: namely, if $\gamma(s)$ is a smooth simple closed curve in $\mathbb{R}^2$ of length 2π, then the area is at most π with equality only for the circle. Without loss, we can suppose γ is arclength parametrized, that is, $\gamma(s) = (x(s), y(s))$, for $0 \le s \le 2\pi$, with

$$|x'(s)|^2 + |y'(s)|^2 = 1 \tag{3.5.113}$$

Thus,

$$\int_0^{2\pi} |x'(s)|^2 + |y'(s)|^2 \, ds = 2\pi \tag{3.5.114}$$

(a) Use Green's formula (see Section 1.4 of Part 3) to prove that

$$\text{Area within } \gamma = \int_0^{2\pi} \tfrac{1}{2} \left(x(s)y'(s) - x'(s)y(s) \right) ds \tag{3.5.115}$$

(b) Expanding x, y in Fourier series and using $|\alpha\beta| \le \frac{1}{2}|\alpha|^2 + |\beta|^2$, prove that

$$\text{Area within } \gamma \le \tfrac{1}{2} \int_0^{2\pi} (|x'(s)|^2 + |y'(s)|^2) \, ds = \pi$$

with equality only if γ is a circle.

Remark. This simple proof of the isoperimetric inequality in dimension 2 is due to Hurwitz [**440**, **441**] in work done in 1901–02. See Groemer–Schneider [**370**] and Groemer [**369**] for results in dimension higher than 2 using spherical harmonic expansions (see Section 3.5 of Part 3). In particular, Groemer [**369**] has many other results on applying Fourier series to geometric inequalities.

27. Prove that a function $f \in C(\partial\mathbb{D})$ is a uniform limit of polynomials in z if and only if $\int_0^{2\pi} e^{in\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} = 0$ for $n = 1, 2, \ldots$. (*Hint*: One direction is already in Proposition 2.4.4; for the other, use Fejér's theorem.)

28. (a) Let P_n be a polynomial of degree n in a complex variable z. Prove that

$$-iP_n^\star(e^{i\theta}) = \int_0^{2\pi} F_n(\theta-\varphi)\, e^{in(\theta-\varphi)} P_n(e^{i\varphi})\, \frac{d\varphi}{2\pi}$$

where

$$F_n(\theta) = \sum_{j=-n+1}^{n-1} (n-|j|)\, e^{ij\theta}$$

and $P_n^\star(e^{i\theta})$ means $\frac{d}{d\theta} f(\theta)$ with $f(\theta) = P_n(e^{i\theta})$ so $|P_n^\star(e^{i\theta})| = |P_n'(e^{i\theta})|$.

(b) Find an explicit formula for F_n (not as a sum) and prove $F_n(\theta) \geq 0$ and $\int F_n(\theta)\, \frac{d\theta}{2\pi} = n$.

(c) Conclude that

$$\sup_{\theta\in[0,2\pi]} |P_n'(e^{i\theta})| \leq n \sup_{\theta\in[0,2\pi]} |P_n(e^{i\theta})|$$

(This is known as *Bernstein's inequality*.)

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Babenko, K. I. **1**: 563, 715; **3**: 125, 692
Bachmann, P. **2A**: 12, 592
Bacry, H. **3**: 401, 402, 692
Báez-Duarte, L. **3**: 161, 692
Baggett, L. **3**: 403, 692
Baik, J. **1**: 630, 715
Baire, R.-L. **1**: 47, 49, 211, 407, 409, 715
Baker, H. F. **4**: 628, 689
Bakonyi, M. **2A**: 305, 593
Bakry, D. **3**: 653, 692
Balian, R. **3**: 402, 692
Banach, S. **1**: 206, 210, 318, 357, 363, 364, 407, 408, 424, 425, 447, 466, 485, 486, 501, 715, 716; **3**: 24, 25, 46, 49, 692, 693; **4**: 43, 100, 689
Bañuelos, R. **3**: 162, 693
Bär, C. **2A**: 21, 593
Bargmann, V. **1**: 538, 716; **3**: 385, 386, 401, 693; **4**: 682, 689
Bari, N. K. **3**: 401, 406, 693
Barndorff-Nielsen, O. E. **1**: 659, 716
Barnes, C. W. **4**: 255, 689
Barut, A. O. **3**: 386, 693
Battle, G. **3**: 402, 405, 433, 434, 693
Bauer, H. **1**: 230, 313, 716; **3**: 177, 276, 693
Baumgärtel, H. **4**: 70, 689
Beals, R. **2A**: 419, 536, 593; **3**: 614, 693; **4**: 231, 255, 689
Bear, H. S. **1**: 230, 716
Beardon, A. F. **2A**: 145, 149, 335, 589, 593; **3**: 127, 693
Beauzamy, B. **1**: 444, 716
Beckenbach, E. F. **2A**: 152, 194, 593
Beckenstein, E. **1**: 443, 706, 746
Beckner, W. **1**: 563, 716; **3**: 335, 652, 693
Begehr, H. G. W. **2A**: 585, 593
Belhoste, B. **2A**: 499, 593
Bell, E. T. **2A**: 404, 593
Bell, S. R. **2A**: 188, 323, 378, 593
Bellman, R. **1**: 486, 716
Beltrami, E. J. **2A**: 562, 593; **4**: 135, 689
Ben-Aroya, A. **3**: 654, 693
Benedetto, J. J. **3**: 333, 693; **4**: 505, 689
Benedicks, M. **3**: 337, 693
Benford, F. **3**: 99, 694
Benguria, R. D. **4**: 605, 690
Bennet, G. **1**: 490
Bennett, A. A. **1**: 364, 716
Bennett, C. **3**: 534, 556, 583, 694
Benyamini, Y. **1**: 357, 716
Bercovici, H. **4**: 322, 722
Berenstein, C. A. **2A**: 469, 593
Berezanskiĭ, Ju. M. **3**: 292, 694
Berezin, F. A. **1**: 538, 716; **3**: 386, 402, 694
Bergh, J. **3**: 556, 583, 694
Bergman, S. **1**: 126, 716, 717; **4**: 489, 690
Berlinet, A. **1**: 126, 717
Bernoulli, D. **1**: 150, 653, 717
Bernoulli, Jakob **1**: 452, 628, 644, 653, 717; **2A**: 396, 437, 516, 533, 593
Bernoulli, Johann **2A**: 393
Bernstein, A. R. **1**: 487, 717
Bernstein, I. N. **1**: 608, 717
Bernstein, S. N. **1**: 78, 82, 717; **2A**: 574; **3**: 291, 694; **4**: 256
Berry, A. C. **1**: 656, 717
Bers, L. **2A**: 38, 230, 367, 593
Berthier, A. M. **3**: 337, 691
Bertoin, J. **1**: 659, 717
Besicovitch, A. S. **1**: 700, 702, 717; **3**: 50, 684, 694; **4**: 367, 419, 690
Besov, O. V. **3**: 583, 694
Bessaga, C. **1**: 357, 717
Bessel, F. W. **1**: 112, 117, 717
Betti, E. **2A**: 403, 593

Beukers, F. **2A**: 398, 593
Beurling, A. **3**: 177, 276, 470, 517, 694; **4**: 128, 367, 369, 385, 627, 690
Bezout, E. **4**: 17, 690
Bialynicki-Birula, I. **3**: 335, 694
Biane, P. **3**: 653, 694
Bieberbach, L. **2A**: 294, 305, 350, 594
Bienaymé, I. J. **1**: 227, 717
Bienvenu, L. **3**: 160, 694
Billingsley, P. **1**: 313, 327, 717; **3**: 79, 125, 694
Binet, J. **2A**: 419, 447, 594
Bing, R. H. **1**: 61, 717
Bingham, N. H. **1**: 645, 717
Birkhoff, G. **1**: 269, 717; **3**: 406, 695
Birkhoff, G. D. **1**: 562; **2A**: 161, 594; **3**: 65, 79–82, 125, 406, 694, 695
Birman, M. **4**: 160, 353, 601, 605, 666, 682, 690
Birnbaum, Z. **1**: 388, 717
Bishop, E. **1**: 13; **3**: 84, 695; **4**: 490, 492, 690, 691
Blachman, N. M. **3**: 652, 695
Blackadar, B. **4**: 314, 691
Blanchard, Ph. **3**: 669, 695
Blankenbecler, R. **4**: 683, 691
Blaschke, W. **1**: 167, 718; **2A**: 455, 594
Blatt, H. P. **1**: 453, 714
Blatter, C. **3**: 433, 695
Bliedtner, J. **3**: 177, 695
Bloch, A. **2A**: 577–579, 594
Bloch, F. **3**: 386, 387, 695
Blumenthal, R. M. **3**: 177, 695
Boas, R. P. **1**: 569, 718
Bôcher, M. **3**: 197, 695
Bochi, J. **3**: 145, 692
Bochner, S. **1**: 126, 341, 512, 564, 718; **2A**: 584, 586, 594; **3**: 543, 603, 695; **4**: 254, 367, 419, 691
Bodineau, T. **1**: 630, 718
Bogachev, V. I. **1**: 230, 686, 718
Bogdan, V. M. **4**: 387, 691
Boggess, A. **3**: 433, 695
Bohl, P. **1**: 487, 718; **3**: 98, 695
Bohman, H. **1**: 83, 718
Bohnenblust, H. F. **1**: 425, 718, 719
Bohr, H. A. **2A**: 118, 420, 594; **3**: 19, 695; **4**: 367, 419, 691
Bokobza, J. **3**: 367, 733
Bolsinov, A. V. **2A**: 479, 594
Boltzman, L. **3**: 79, 80, 695
Bolzano, B. **1**: 73, 156, 718
Bombelli, R. **2A**: 4, 304, 594
Bonahon, F. **2A**: 333, 594
Bonami, A. **3**: 337, 652, 695
Bonet, J. **1**: 443, 748
Bonsall, F. **4**: 357, 691
Boole, G. **3**: 513, 696
Boon, M. **3**: 402, 696
Borchardt, C. W. **2A**: 87, 497, 594
Border, K. **1**: 443, 485, 714, 718
Borel, A. **4**: 443, 691
Borel, É. **1**: 73, 211, 228, 568, 628, 644, 645, 656, 718; **2A**: 64, 94, 182, 469, 577, 578, 594; **3**: 97, 696; **4**: 266, 691
Borodin, A. N. **1**: 327
Borsuk, K. **1**: 487, 719
Borwein, D. **4**: 506, 691
Borzov, V. V. **4**: 605, 690
Bosma, W. **3**: 125, 696
Bott, R. **1**: 607, 715; **4**: 217, 689
Bottazzini, U. **2A**: 3, 36, 475, 517, 594
Böttcher, A. **4**: 218, 691
Bouligand, G. **1**: 702, 719; **3**: 231, 696
Bouniakowsky, V. **1**: 117, 719
Bouquet, J.-C. **2A**: 87, 130, 595
Bourbaki, N. **1**: 48, 74, 99, 102, 106, 125, 225, 230, 350, 443, 447, 501, 706, 719; **2A**: 57, 595
Bourdon, P. **3**: 177, 692
Bourgain, J. **1**: 365, 719; **3**: 49, 84, 85, 682, 683, 685, 696
Bowen, R. **3**: 126, 696
Bowman, F. **2A**: 477, 595
Bradford, S. C. **1**: 658, 719
Branquinho, A. **4**: 255, 711
Brascamp, H. J. **1**: 394, 563, 719; **3**: 563, 696
Bratteli, O. **3**: 433, 696; **4**: 314, 691
Brauer, R. **2A**: 305
Breiman, L. **1**: 617, 719
Brelot, M. **3**: 177, 231, 273, 274, 276, 696, 697
Brenke, W. C. **4**: 254, 692
Bressoud, D. M. **1**: 193, 203, 225, 228, 719
Breuer, J. **2A**: 58, 59, 241, 242, 595
Brezinski, C. **2A**: 304, 595
Brézis, H. **1**: 249, 719
Brezis, H. **3**: 336, 697
Brieskorn, E. **2A**: 267, 595

Brillhart, J. **4**: 370, 692
Briot, Ch. **2A**: 87, 130, 595
Brocot, A. **2A**: 333, 595
Brodskiĭ, M. S. **4**: 128, 692
Bros, J. **1**: 539, 720
Brouncker, W. **2A**: 282, 304
Brouwer, L. E. J. **1**: 13, 486, 575, 701, 720; **2A**: 164, 595
Browder, A. **2A**: 157, 595; **4**: 489, 692
Browder, F. **2A**: 26
Brown, G. **1**: 582, 720
Brown, J. L. **1**: 720
Brown, J. R. **3**: 65, 697
Brown, L. G. **4**: 199, 692
Brown, R. **1**: 326, 720
Brown, R. F. **1**: 485, 720
Bruhat, F. **1**: 513, 720
Bugeaud, Y. **2A**: 304, 595
Bullen, P. S. **1**: 388, 720
Burckel, R. B. **2A**: 165, 323, 595
Bürgisser, P. **2A**: 112, 595
Burkholder, D. L. **3**: 25, 162, 697
Burkinshaw, O. **1**: 269, 714
Burns, A. **2A**: 48, 595
Burnside, W. **4**: 443, 692
Busemann, H. **3**: 48, 697
Butera, P. **3**: 401, 693
Buttazzo, G. **1**: 453, 720
Butzer, P. L. **1**: 569, 575, 720; **2A**: 443, 595

Calabi, E. **2A**: 398, 593
Calderón, A.-P. **2A**: 177, 595; **3**: 36, 83, 276, 387, 542, 601, 614, 697; **4**: 69, 688
Calkin, J. W. **3**: 581, 697; **4**: 152, 198, 568, 692
Callahan, J. J. **3**: 17, 697
Calvin, C. **3**: 387, 702
Campbell, J. E. **4**: 628, 692
Campiti, M. **1**: 83, 714
Candès, E. J. **3**: 339, 698
Cantelli, F. P. **1**: 644, 720, 721
Cantero, M. J. **4**: 284, 692
Cantor, G. **1**: 9, 13, 16, 47, 49, 50, 201, 228, 721; **2A**: 404
Carathéodory, C. **1**: 464, 564, 686, 721; **2A**: 94, 117, 200, 238, 239, 314, 315, 323, 324, 455, 578, 583, 595, 596; **4**: 321, 692
Carbery, A. **3**: 684, 698
Cardano, G. **2A**: 4
Carey, A. L. **3**: 387, 698
Carl, S. **1**: 485, 721
Carleman, T. **1**: 433, 721; **4**: 160, 186, 192, 534, 692
Carlen, E. A. **3**: 652, 653, 698
Carleson, L. **1**: 153, 721; **2A**: 194, 596; **3**: 172, 698; **4**: 389, 490, 692
Carmona, R. **1**: 329, 721; **3**: 294, 653, 698
Carroll, L. **1**: 1, 721; **4**: 430, 686, 692, 693
Cartan, É. **4**: 446, 693
Cartan, H. **1**: 48, 101, 350, 721; **2A**: 57, 239, 247, 568, 574, 584, 585, 596; **3**: 274, 276, 698
Casanova, G. **3**: 160
Casher, A. **4**: 218, 687
Casorati, F. **2A**: 128, 596
Cassels, J. W. S. **2A**: 477, 596
Cataldi, P. **2A**: 304
Cauchy, A-L. **1**: 6, 26, 32, 47, 49, 112, 117, 193, 227, 388, 486, 630, 655, 721, 722; **2A**: 37, 39, 47, 49, 68, 86, 87, 100, 180, 214, 332, 457, 499, 569, 596; **4**: 17, 693
Cavalieri, B. **1**: 288, 722
Cayley, A. **1**: 26, 722; **4**: 17, 443, 693
Čech, E. **1**: 101, 722; **4**: 412, 693
Cellérier, Ch. **1**: 156, 722
Chacon, R. V. **3**: 86, 698
Chae, S. B. **1**: 225, 228, 722
Champernowne, D. G. **3**: 97, 698
Chandler, R. E. **4**: 412, 693
Chandrasekharan, K. **1**: 230, 574, 722; **2A**: 477, 597
Chang, Y-C. **3**: 684, 701
Chapman, R. **2A**: 215, 597
Chattopadhyay, A. **4**: 353, 693
Chavel, I. **4**: 605, 693
Chebyshev, P. **2A**: 305
Chebyshev, P. L. **1**: 227, 433, 628, 644, 653, 655, 722; **4**: 241, 266, 693
Cheeger, J. **2A**: 21, 597
Cheema, M. S. **2A**: 535, 597
Chemin, J.-Y. **3**: 585, 698
Chen, J. **2A**: 152, 597
Cheney, E. W. **4**: 267, 693
Chernoff, P. R. **1**: 102, 722; **4**: 629, 693
Chevalley, C. **1**: 48, 102, 722
Chihara, T. S. **4**: 231, 693
Cho, Y. **3**: 172, 698

Cholesky, A-L. **1**: 135
Choquet, G. **1**: 464, 465, 566, 722, 723; **3**: 274, 698
Choquet-Brohat, Y. **2A**: 12, 597
Chousionis, V. **3**: 603, 698
Chow, Y. S. **1**: 617, 723
Christ, M. **3**: 172, 603, 684, 698, 699
Christensen, O. **3**: 401, 699
Christiansen, J. **4**: 267, 693
Christoffel, E. B. **2A**: 351, 597; **3**: 291, 699
Chung, K. L. **1**: 327, 617, 674, 723; **3**: 155, 699
Ciesielski, K. **1**: 14, 327, 723
Cima, J. A. **2A**: 188, 597; **3**: 489, 699
Clarke, F. H. **3**: 652, 691
Clarkson, J. A. **1**: 388, 444, 723
Clausen, M. **2A**: 112, 595
Clebsch, A. **2A**: 518, 597
Coddington, E. A. **4**: 569, 693
Cohen, P. J. **1**: 13, 723
Cohn, D. L. **1**: 230, 723
Coifman, R. R. **3**: 433, 534, 535, 614, 699, 719, 720
Collatz, L. **1**: 675, 723
Conlon, J. G. **3**: 669, 699
Constantinescu, C. **3**: 177, 699; **4**: 44, 314, 694
Constantinescu, T. **2A**: 305, 593, 597
Conway, J. B. **2A**: 323, 324, 468, 579, 597
Cooley, J. W. **1**: 155, 723
Copeland, A. H. **3**: 97, 699
Copson, E. T. **2A**: 214, 597
Cordes, H. O. **3**: 367, 614, 699; **4**: 534, 694
Córdoba, A. **1**: 539, 723; **3**: 48, 685, 699
Corduneanu, C. **4**: 419, 694
Cornea, A. **3**: 177, 699
Cornu, A. **2A**: 214
Cotes, R. **2A**: 59, 597
Cotlar, M. **3**: 83, 542, 613, 699
Coulomb, J. **1**: 48
Courant, R. **1**: 606, 723; **2A**: 57, 323, 597, 605; **3**: 17, 699; **4**: 109, 118, 603, 694
Cowling, M. **1**: 563, 723
Cox, D. A. **2A**: 533, 597
Craig, W. **2A**: 377, 597; **3**: 291, 295, 297, 699
Cramér, H. **1**: 654, 656, 666, 723
Cramer, G. **4**: 17, 694
Crépel, P. **3**: 160, 699
Croft, H. T. **3**: 49, 50, 700
Cronin, J. **1**: 485, 487, 724
Crowdy, D. **2A**: 351, 597
Crum, M. M. **4**: 129, 694
Crummett, W. P. **1**: 150, 762
Császár, A. **4**: 255, 694
Curtis, C. W. **4**: 443, 446, 694
Cwikel, M. **3**: 534, 669, 700; **4**: 161, 694
Cycon, H. L. **3**: 294, 700; **4**: 217, 218, 538, 694

d'Alembert, J. **2A**: 37, 87, 597
da Silva Dias, C. **2A**: 230, 598
Dacorogna, B. **1**: 453, 724
Dahlberg, B. E. J. **3**: 274, 700
Dajani, K. **3**: 123, 700
d'Alembert, J. **1**: 150, 606, 724
Dalzell, D. P. **1**: 724
Damanik, D. **3**: 293, 700
Damelin, S. B. **3**: 401, 700
Daniell, P. J. **1**: 229, 269, 724
Darboux, J-G. **1**: 74; **2A**: 165, 438, 574, 598; **3**: 291, 700
Daston, L. **1**: 628, 724
Daubechies, I. **3**: 401, 403, 433, 434, 700
Dauben, J. **1**: 16, 724
David, G. **3**: 602, 700
Davidson, K. **4**: 314, 695
Davies, E. B. **3**: 336, 622, 650, 653, 700, 701; **4**: 323, 601, 632, 695
Davis, B. **2A**: 556, 598; **3**: 162, 514, 693, 701
Davis, C. **4**: 218, 695
Davis, K. M. **3**: 684, 701
Davis, P. J. **2A**: 405, 421, 598
Day, M. M. **1**: 444, 724
de Boor, C. **4**: 695
de Branges, L. **2A**: 89; **4**: 353, 666, 695
de Bruijn, N. G. **3**: 99, 701
de Guzmán, M. **3**: 25, 701
de la Vallée Poussin, Ch. J. **1**: 82, 161, 162, 725; **2A**: 469; **3**: 64, 701; **4**: 267, 695
de Leeuw, K. **3**: 472, 701; **4**: 490, 691
de Moivre, A. **2A**: 437, 598
de Moor, B. **4**: 135, 712
de Snoo, H. S. V. **4**: 601, 666, 702
de Wolf, R. **3**: 654, 693
Deans, S. R. **1**: 548, 724
de Branges, L. **1**: 466, 724

Dedekind, R. **1**: 9, 724; **2A**: 315
De Giorgi, E. **1**: 453, 724
Deift, P. A. **1**: 630, 715, 724; **2A**: 152; **4**: 57, 218, 695
de Jonge, E. **1**: 269, 724
Del Pino, M. **3**: 582, 701
del Rio, R. **1**: 702, 725; **3**: 514, 701; **4**: 344, 695
Dellacherie, C. **3**: 161, 701
Delort, J.-M. **1**: 539, 725
Delsarte, J. **1**: 48
Demange, B. **3**: 337, 695
Demengel, F. **3**: 583, 701
Demengel, G. **3**: 583, 701
de Moivre, A. **1**: 628, 653, 654, 656, 725
de Monvel, B. **1**: 514
Denisov, S. A. **3**: 293, 701
Denjoy, A. **1**: 230, 725; **2A**: 152, 194, 598; **3**: 99, 701
Denker, J. **3**: 373, 701
Deny, J. **3**: 177, 274, 276, 694, 698, 701; **4**: 627, 690, 695
Deprettere, E. F. **4**: 135, 695
de Rham, G. **1**: 117, 164, 512, 513, 538, 725; **2A**: 26, 598
Derriennic, Y. **3**: 145, 702
Desargues, G. **2A**: 282, 598
Descartes, R. **1**: 26, 725
Deuschel, J-D. **3**: 652, 654, 702
DeVore, R. A. **1**: 84, 725; **3**: 534, 694
DeWitt-Morette, C. **2A**: 12, 597
Diaconis, P. **3**: 653, 702; **4**: 443, 695
Diamond, F. **2A**: 550, 598
DiBenedetto, E. **2A**: 12, 598; **3**: 50, 702
Dienes, P. **2A**: 57, 598
Dieudonné, J. **1**: 48, 350, 443, 458, 487, 501, 711, 712, 725; **2A**: 239; **4**: 217, 695
Dillard-Bleick, M. **2A**: 12, 597
Dineen, S. **2A**: 585, 598
Dinghas, A. **2A**: 57, 598
Dini, U. **1**: 138, 152, 202, 226, 228, 231, 486, 725
Dirac, P. A. M. **1**: 725
Dirichlet, P. G. **1**: 68, 140, 150, 151, 228, 726; **2A**: 87, 304, 315, 598; **3**: 273
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Ditzian, Z. **1**: 84, 726
Dixmier, J. **4**: 314, 695
Dixon, A. C. **4**: 41, 56, 695
Dixon, J. D. **2A**: 143, 598
Dobrushin, R. **1**: 629
Doeblin, W. **1**: 658, 726; **3**: 124, 702
Doetsch, G. **2A**: 177, 598
Dolbeault, J. **3**: 582, 701
Dollard, J. D. **1**: 607, 726
Dominici, D. **2A**: 437, 598
Donaldson, S. **2A**: 589, 599
Donoghue, W. F. **4**: 128, 343, 353, 606, 688, 696
Donoho, D. **3**: 339, 702
Donsker, M. D. **1**: 328, 726
Doob, J. L. **1**: 230, 327, 656, 726; **3**: 84, 160, 161, 165, 177, 276, 702
Doran, R. S. **4**: 428, 696
Douglas, R. G. **4**: 199, 218, 692, 696
Driscoll, T. A. **2A**: 350, 351, 599
Du Val, P. **2A**: 477, 599
du Bois-Reymond, P. **1**: 14, 49, 152, 201, 726
Dudley, R. M. **1**: 239, 313, 617, 726
Duffin, R. J. **3**: 401, 403, 702
Duflo, M. **3**: 387, 702
Dufresnoy, J. **4**: 129, 696
Dugac, P. **1**: 74, 726
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Duistermaat, J. J. **3**: 350, 367, 702; **4**: 255, 628, 696
Dummit, D. S. **2A**: 8, 599
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Dunford, N. **1**: 275, 487, 726; **2A**: 88, 599; **3**: 86, 702; **4**: 69, 186, 192, 568, 569, 696
Dunham, W. **2A**: 88, 395, 599
Dunnington, G. W. **2A**: 29, 599
Duoandikoetxea, J. **1**: 149, 726
Duran, A. **4**: 255, 696
Duren, P. **3**: 439, 464, 513, 702
Durrett, R. **1**: 327, 617, 726; **3**: 162, 163, 702
Dvir, Z. **3**: 685, 702
Dvoretzky, A. **1**: 328, 726
Dym, H. **1**: 126, 537, 574, 726, 727; **3**: 337, 702
Dynkin, E. B. **1**: 629, 674, 727
Dzhuraev, A. **2A**: 585, 593

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Ebbinghaus, H.-D. **2A**: 397, 599
Eberlein, W. F. **2A**: 397, 477, 479, 536, 592, 599
Ebin, D. G. **2A**: 21, 597

Eckart, C. **4**: 135, 696
Edgar, G. **1**: 700, 727
Edmunds, D. E. **4**: 198, 602, 696
Effros, E. G. **4**: 218, 696
Eggleston, H. G. **1**: 387, 727
Egorov, D. **1**: 226, 249, 727
Egorov, Y. V. **3**: 367, 368, 703
Ehrenfest, P. **3**: 80, 703
Ehrenfest, T. **3**: 80, 703
Ehrenpreis, L. **1**: 607, 727; **2A**: 584, 599
Ehresmann, C. **1**: 48
Eidelheit, M. **2A**: 405, 599
Eilenberg, S. **2A**: 26, 599
Einsiedler, M. **3**: 79, 123, 126, 703
Einstein, A. **1**: 326, 606, 727; **2A**: 266
Eisenstein, G. **2A**: 315, 517, 599
Ekholm, T. **3**: 669, 703
Emerson, R. W. **1**: 1, 727; **2A**: 1, 599; **3**: 1, 703; **4**: 1, 697
Émery, M. **3**: 652, 653, 692, 703
Enderton, H. B. **1**: 14, 727
Enflo, P. **1**: 488, 727; **4**: 100, 697
Engel, F. **4**: 628, 710
Enss, V. **4**: 321, 697
Epple, M. **1**: 49, 727
Epstein, B. **2A**: 468, 603
Epstein, D. B. A. **2A**: 324, 599
Erdős, L. **4**: 218, 697
Erdos, J. A. **4**: 128, 187, 697
Erdős, P. **1**: 153, 328, 646, 726, 727; **3**: 97, 291, 292, 699, 703
Erlang, A. K. **1**: 666, 727
Ermenko, A. **2A**: 574, 578, 599, 600; **3**: 218, 703
Eskin, G. I. **3**: 368, 703
Esseen, C. G. **1**: 656, 727
Essén, M. **3**: 177, 488, 691, 703
Estermann, T. **2A**: 101, 600
Euclid **2A**: 306, 600
Euler, L. **1**: 26, 47, 150, 727, 728; **2A**: 4, 59, 87, 214, 215, 222, 255, 282, 304, 393–395, 419, 438, 517, 533, 600
Evans, G. C. **3**: 273, 274, 703
Evans, G. W. **1**: 728
Evans, L. C. **1**: 453, 606, 700, 728
Evans, W. D. **4**: 198, 602, 696
Ewald, P. **1**: 567, 728
Ewing, G. M. **1**: 728
Exner, P. **4**: 630, 697
Faber, G. **2A**: 58, 600; **3**: 291, 703; **4**: 268, 697
Fabes, E. B. **3**: 653, 704
Fabian, M. **1**: 357, 444, 728
Fabry, E. **2A**: 58, 600
Fagnano, C. G. **2A**: 516, 600
Falconer, K. **1**: 156, 700, 702, 728
Fan, K. **1**: 630, 728; **4**: 135, 697
Farey, J. **2A**: 332, 601
Faris, W. G. **3**: 654, 704; **4**: 601, 632, 697
Farkas, H. M. **2A**: 267, 533, 534, 589, 601, 615; **3**: 316, 704
Fatou, P. **1**: 161, 226, 249, 728; **2A**: 180; **3**: 59, 704
Favard, J. **4**: 241, 697
Federbush, P. **3**: 651, 704
Federer, H. **1**: 700, 728
Fefferman, C. **1**: 539, 723; **3**: 48, 172, 336, 498, 514, 534, 603, 669, 682, 684, 685, 704; **4**: 228, 697
Fefferman, R. **3**: 48, 699
Feichtinger, H. G. **3**: 390, 704
Fejér, L. **1**: 82, 139, 142, 152, 153, 728; **2A**: 315, 455, 596; **3**: 434, 704; **4**: 285, 321, 697
Fekete, M. **4**: 268, 697
Feller, W. **1**: 617, 656, 657, 728; **3**: 48, 697
Fermat, P. **1**: 628; **2A**: 518
Fermi, E. **1**: 453, 728
Ferreira, P. J. S. G. **1**: 569, 720
Feynman, R. P. **1**: 588, 728; **4**: 27, 630, 697
Figalli, A. **3**: 654, 704
Figotin, A. **3**: 294, 723
Fillmore, P. A. **4**: 199, 692
Finch, S. R. **2A**: 579, 601
Findley, E. **3**: 292, 704
Fischer, E. **1**: 150, 153, 226, 728; **4**: 109, 604, 697
Fischer, G. **2A**: 267, 601
Fischer, H. **1**: 654, 728
Fisher, S. D. **2A**: 378, 601
Flandrin, P. **3**: 433, 704
Fock, V. **1**: 538, 729
Foias, C. **4**: 322, 722
Fokas, A. S. **2A**: 351, 591
Folland, G. B. **1**: 149, 538, 606, 729; **3**: 333, 338, 342, 704
Fomenko, A. T. **2A**: 479, 594

Fomin, S. **1**: 629
Foote, R. M. **2A**: 8, 599
Ford, J. W. M. **4**: 83, 697
Ford, L. R. **2A**: 282, 289, 304, 333, 335, 601; **3**: 127, 705
Formin, S. V. **4**: 603, 707
Forster, O. **2A**: 266, 267, 589, 601
Fourier, J. **1**: 150, 151, 546, 607, 729; **2A**: 499
Fournier, J. J. F. **1**: 563, 729; **3**: 583, 691
Fox, L. **4**: 266, 697
Fröhlich, J. **1**: 608, 730
Fraenkel, A. A. **1**: 13, 729
Frank, R. L. **3**: 564, 669, 670, 703, 705
Franks, J. **1**: 230, 729
Fréchet, M. **1**: 6, 47, 49, 60, 61, 74, 75, 118, 125, 229, 363–365, 501, 630, 729; **2A**: 470; **3**: 40, 705
Fredholm, I. **1**: 47, 729; **2A**: 601; **4**: 41, 99, 182, 697, 698
Fremlin, D. H. **1**: 269, 729
Fresnel, A. **2A**: 214
Freudenthal, H. **1**: 269, 485, 487, 729; **4**: 600, 698
Freund, G. **3**: 292, 705
Friedman, A. **1**: 606, 730
Friedrichs, K. O. **1**: 512; **3**: 367, 581, 705; **4**: 27, 600, 698
Frink, O. **1**: 102, 722; **4**: 254, 603, 698, 708
Fristedt, B. **1**: 313, 617, 730; **3**: 162, 705
Fritzsche, K. **2A**: 585, 602
Frobenius, F. G. **2A**: 404, 568
Frobenius, G. **1**: 675, 730; **4**: 18, 387, 444, 445, 698
Froese, R. G. **3**: 294, 700; **4**: 217, 218, 538, 694
Frostman, O. **3**: 273, 274, 276, 705
Fubini, G. **1**: 288, 730
Fuchs, L. **2A**: 568
Fuglede, B. **2A**: 424, 601
Fukamiya, M. **4**: 428, 698
Fukushima, M. **3**: 177, 276, 705
Fulton, W. **2A**: 23, 601; **4**: 443, 698
Füredi, Z. **3**: 50, 705
Furi, M. **1**: 485, 720
Furstenberg, H. **1**: 51, 730; **3**: 84, 123, 145, 146, 705, 706

Gabor, D. **3**: 334, 386, 401, 706
Gagliardo, E. **3**: 582, 681, 706
Gaier, D. **2A**: 450, 609
Galois, É **2A**: 499
Galton, F. **1**: 648, 730
Gamelin, T. W. **2A**: 157, 367, 601; **3**: 316, 706; **4**: 389, 489, 490, 698
Gantmacher, F. R. **1**: 675, 730; **2A**: 85, 601
Garban, C. **3**: 650, 706
Gardiner, S. J. **2A**: 137, 601; **3**: 177, 692
Gårding, L. **1**: 513, 607, 715, 762; **2A**: 167, 173, 400, 601
Gariepy, R. F. **1**: 700, 728
Garling, D. J. H. **1**: 387, 615, 730; **3**: 161, 166, 547, 601, 650, 706; **4**: 187, 698
Garnett, J. B. **2A**: 378, 601; **3**: 274, 439, 534, 706; **4**: 389, 490, 698
Garsia, A. M. **3**: 49, 50, 83, 86, 91, 161, 706
Gateaux, R. **1**: 365, 730
Gauss, C. F. **1**: 26, 567, 628, 653, 730; **2A**: 20, 37, 87, 315, 419, 517, 533, 602; **3**: 124, 197, 273, 706; **4**: 17, 698
Gay, R. **2A**: 469, 593
Gel'fand, I. M. **1**: 341, 513, 538, 548, 629, 730; **3**: 402, 706; **4**: 56, 57, 69, 128, 357, 387, 388, 399, 405, 406, 428, 447, 467, 489, 504, 698, 699
Gençay, C. **3**: 433, 706
Georgescu, V. **4**: 321, 666, 688, 699
Geronimus, Ya. L. **2A**: 306, 602; **4**: 231, 282–284, 699
Gesztesy, F. **4**: 87, 344, 352, 353, 602, 666, 689, 699
Getoor, R. K. **3**: 177, 695
Getzler, E. **4**: 217, 699
Gibbs, J. W. **1**: 148, 156, 157, 387, 730
Gilbard, D. **1**: 606, 731
Gilbarg, D. **3**: 177, 276, 706
Gilbert, D. J. **4**: 570, 699
Gilkey, P. B. **4**: 217, 699
Gillman, L. **4**: 412, 700
Gilmore, R. **3**: 386, 706
Ginibre, J. **3**: 683, 706
Giradello, L. **3**: 386, 401, 693
Girondo, E. **2A**: 589, 602
Glaisher, J. W. L. **2A**: 536, 602
Glasner, E. **3**: 99, 706

Glauber, R. J. **3**: 385, 386, 707
Gleason, A. M. **4**: 392, 490, 700
Glicksberg, I. **2A**: 101, 602; **4**: 490, 700
Glimm, J. **1**: 329, 731; **3**: 651, 654, 656, 707; **4**: 428, 700
Gnedenko, B. V. **1**: 658, 659, 731
Godement, R. **2A**: 568, 602; **3**: 386, 707; **4**: 468, 504, 700
Goebel, K. **1**: 485, 731
Goh'berg, I. C. **3**: 603, 707; **4**: 134, 152, 153, 187, 192, 217, 218, 601, 700
Goldberg, M. **3**: 683, 707
Goldberg, S. **4**: 187, 700
Goldberger, M. L. **4**: 683, 691
Goldstine, H. H. **1**: 444, 731
Golub, G. H. **4**: 135, 700
Gomez-Ullate, D. **4**: 255, 700
Gonçalves, P. **1**: 702, 713
González-Diez **2A**: 589, 602
Gordon, A. **3**: 296, 707
Gordon, A. Ya. **4**: 344, 700
Gordon, C. **4**: 605, 700
Górniewicz, L. **1**: 485, 720
Gosset, W. S. **1**: 666
Goursat, E. **1**: 485, 486, 731; **2A**: 68, 602
Gowers, W. T. **4**: 44, 100, 700, 701
Grätzer, G. **1**: 11, 731
Graev, M. I. **1**: 513, 548, 730
Grafakos, L. **1**: 149, 731; **3**: 534, 535, 603, 682, 684, 707
Graham, L. **1**: 249, 731
Graham, R. L. **2A**: 333, 602
Gram, J. P. **1**: 132, 134, 731
Granas, A. **1**: 485, 731
Grassmann, H. **1**: 9, 731
Grattan-Guiness, I. **1**: 150, 372, 731
Grauert, H. **2A**: 585, 602
Gray, J. **1**: 37, 731; **2A**: 36, 68, 117, 314, 335, 475, 517, 594, 602
Gray, L. **1**: 313, 617, 730; **3**: 162, 705
Green, B. **3**: 683, 707
Green, G. **1**: 606, 731; **2A**: 315, 602; **3**: 197, 273, 707
Greene, R. E. **2A**: 156, 468, 602
Greenleaf, A. **3**: 682, 707
Greenleaf, F. P. **1**: 486, 731
Griffiths, P. A. **2A**: 267, 589, 602
Grimmett, G. **1**: 617, 731
Gröchenig, K. **3**: 403, 707
Groemer, H. **1**: 167, 731, 732
Grolous, J. **1**: 387, 732
Gronwall, T. H. **1**: 732
Gross, L. **3**: 650, 652–654, 656, 701, 707, 708
Grossmann, A. **3**: 368, 386, 387, 401, 402, 692, 700, 708
Grosswald, E. **2A**: 222, 615
Grothendieck, A. **1**: 182, 413, 514, 732; **2A**: 230, 602; **4**: 144, 186, 701
Grubb, G. **4**: 601, 602, 701
Grümm, H. R. **4**: 153, 701
Grünbaum, F. A. **4**: 255, 696, 701
Guckenheimer, J. **3**: 99, 708
Gudermann, Chr. **2A**: 440, 602
Guggenheimer, H. **2A**: 21, 603
Guionnet, A. **3**: 622, 650, 654, 708
Gundy, R. F. **3**: 162, 697
Gunning, R. C. **2A**: 585, 589, 603
Guo, B.-N. **2A**: 447, 622
Gustafson, K. **4**: 198, 701
Güttinger, P. **4**: 27, 701
Gvishiani, A. D. **4**: 217, 707

Haar, A. **1**: 350, 732; **3**: 434, 708; **4**: 267, 701
Habala, P. **1**: 357, 444, 728
Hacking, I. **1**: 628, 732
Hadamard, J. **1**: 75, 238, 487, 500, 501, 512, 601, 607, 732; **2A**: 50, 58, 118, 177, 419, 430, 468–470, 574, 603; **3**: 437, 582, 708
Hahn, H. **1**: 205, 269, 364, 408, 424, 732; **4**: 254, 313, 701
Hahn, L-S. **2A**: 468, 603
Hahn, W. **4**: 255, 701
Haine, L. **4**: 255, 701
Hairer, E. **2A**: 305, 603
Hájek, P. **1**: 357, 444, 728
Hales, T. C. **2A**: 164, 603
Hall, B. C. **4**: 628, 701
Halmos, P. R. **1**: 13, 211, 230, 257, 732; **3**: 79, 517, 708; **4**: 218, 299, 313, 314, 536, 701
Hamburger, H. **1**: 433, 732; **4**: 658, 701
Hamilton, W. R. **4**: 17, 702
Hammersley, J. M. **3**: 145, 708
Han, Q. **3**: 177, 708
Hanche-Olsen, H. **2A**: 75, 603
Hancock, H. **2A**: 477, 536, 603
Handscomb, D. C. **4**: 266, 711
Hankel, H. **1**: 9, 202, 228, 732, 733
Hanner, O. **1**: 388, 733

Hansen, W. **3**: 177, 695
Hardy, G. H. **1**: 156, 394, 569, 582, 645, 733; **2A**: 12, 603; **3**: 36, 46, 52, 98, 213, 335, 337, 444, 458, 464, 487, 488, 557, 559, 562, 564, 709; **4**: 367, 505, 506, 605, 702
Harish-Chandra **1**: 513, 733
Harnack, A. **1**: 228; **3**: 198, 709
Haros, C. **2A**: 332, 603
Haroske, D. **3**: 583, 709
Harriot, T. **2A**: 272
Harris, J. **2A**: 267, 589, 602; **4**: 443, 698
Hartman, P. **1**: 645, 733; **3**: 83, 536, 709
Hartogs, F. **2A**: 583, 603; **3**: 213, 709; **4**: 489, 702
Hasselblatt, B. **3**: 84, 713
Hassi, S. **4**: 601, 666, 702
Hatch, D. **2A**: 335, 604
Hatcher, A. **1**: 487, 733; **2A**: 23, 24, 26, 142, 165, 604
Hausdorff, F. **1**: 35, 47, 48, 50, 60, 61, 210, 336, 364, 563, 645, 700, 733; **4**: 628, 702
Havil, J. **2A**: 420, 421, 604
Havin, V. **3**: 333, 709
Havinson, S. Ya. **2A**: 378, 604
Hawkins, T. **1**: 225, 228, 733; **4**: 443, 446, 702
Hayes, B. **2A**: 333, 604
Hayman, W. K. **3**: 177, 253, 709
Haynsworth, E. V. **4**: 208, 702
Heaviside, O. **1**: 512, 733
Hedlund, G. A. **3**: 125, 709
Heikkilä, S. **1**: 485, 721
Heil, C. **3**: 401, 403, 434, 710; **4**: 100, 702
Heilbronn, H. **2A**: 152, 604
Heine, E. **1**: 9, 68, 73, 201, 228, 733
Heins, M. **2A**: 292, 604
Heinz, E. **4**: 606, 703
Heisenberg, W. **3**: 333, 710
Helemskii, A. Ya. **4**: 44, 703
Helgason, S. **1**: 548, 733
Hellegouarch, Y. **2A**: 533, 604
Hellinger, E. **1**: 408, 413, 433, 733, 734; **4**: 299, 313, 703
Hellman, H. **4**: 27, 703
Hellwig, G. **1**: 606, 734
Helly, E. **1**: 238, 363, 364, 408, 424–426, 447, 734
Helmer, O. **2A**: 406, 604
Helms, L. L. **1**: 453, 734; **3**: 177, 710
Helson, H. **4**: 489
Hempel, R. **4**: 603, 703
Henderson, R. **1**: 387, 734
Henrici, P. **2A**: 57, 604
Hensley, D. **2A**: 304, 604; **3**: 123, 125, 710
Henstock, R. **1**: 230, 734
Herbert, D. **3**: 291, 710
Herbst, I. W. **3**: 564, 710; **4**: 572, 703
Herglotz, G. **1**: 564, 565, 734; **2A**: 239, 394, 604; **3**: 463, 513, 710
Herival, J. **1**: 151, 734
Hermes, H. **2A**: 397, 599
Hermite, Ch. **1**: 26, 137, 175, 193, 734; **2A**: 227, 305, 400, 479, 499, 568, 574, 604
Hernández, E. **1**: 519, 734; **3**: 433, 710
Herz, C. **3**: 684, 710
Hess, H. **4**: 627, 628, 703
Hewitt, E. **1**: 157, 734; **4**: 443, 468, 504, 505, 703
Hewitt, R. E. **1**: 157, 734
Hida, H. **2A**: 550, 604
Higgins, J. R. **1**: 568, 569, 720, 734
Hilbert, D. **1**: 15, 17, 59, 117, 125, 371, 387, 447, 514, 606, 723, 734; **2A**: 157, 238, 246, 266, 368, 604; **3**: 273, 316, 487, 710; **4**: 41, 42, 56, 99, 108, 118, 192, 299, 603, 694, 703
Hildebrandt, T. H. **4**: 43, 703
Hilden, H. M. **4**: 118
Hill, G. W. **4**: 41, 703
Hille, E. **2A**: 12, 157, 404, 605; **4**: 99, 192, 703
Hirschman, I. I. **3**: 335, 710
Hirzebruch, F. **2A**: 397, 599
Hockman, M. **2A**: 333, 605
Høegh-Krohn, R. **1**: 290, 734; **3**: 652, 729
Hoffman, K. **4**: 357, 489, 703
Hölder, E. **1**: 372, 387, 734
Hölder, O. **2A**: 404, 419, 605
Hollenbeck, B. **3**: 489, 710
Holmes, P. **3**: 99, 708
Hopf, E. **3**: 81–83, 89, 91, 125, 710
Hopf, H. **1**: 48, 60, 106, 714; **2A**: 20, 26, 605
Hörmander, L. **1**: 513, 606, 608, 734, 735; **2A**: 400, 585, 601, 605; **3**:

213, 350, 366, 367, 370, 603, 613, 683, 691, 702, 710, 711; **4**: 667, 703
Horn, A. **1**: 394, 735
Horváth, J. **1**: 443, 735; **3**: 614, 711
Howard, P. **1**: 13, 735
Howe, R. **3**: 336, 368, 614, 711
Hrŭščev, S. V. **3**: 514, 711
Hubbard, B. B. **3**: 434, 711
Humphreys, J. E. **4**: 443, 704
Hundertmark, D. **3**: 669, 670, 711; **4**: 603, 628, 683, 685, 704
Hunt, G. A. **3**: 177, 276, 711
Hunt, R. A. **1**: 153, 735; **3**: 172, 556, 711
Hunzicker, W. **4**: 666, 704
Hurewicz, W. **2A**: 26, 605
Hurwitz, A. **1**: 167, 350, 735; **2A**: 57, 246, 304, 404, 605
Husemoller, D. **2A**: 477, 518, 605
Husimi, K. **3**: 386, 712
Huxley, M. N. **4**: 605, 704
Huygens, C. **1**: 607, 628, 735
Hwang, I. L. **3**: 614, 712

Iagolnitzer, D. **1**: 539, 720
Ichinose, T. **4**: 630, 704
Iftimovici, A. **4**: 666, 699
Igusa, J. **2A**: 534, 605
Ikehara, S. **4**: 506, 704
Ince, E. L. **2A**: 12, 605
Indrei, E. **3**: 654, 712
Ingham, A. E. **2A**: 214, 605; **4**: 504, 513, 704
Ionescu Tulcea, A. **3**: 161, 683, 712
Ionescu Tulcea, C. **3**: 161, 712
Iosevich, A. **3**: 682, 712
Iosifescu, M. **2A**: 304, 605; **3**: 123, 712
Isaacs, M. I. **4**: 443, 704
Ishii, K. **3**: 294, 712
Ismail, M. E. H. **1**: 135, 735; **4**: 231, 255, 704
Issac, R. **3**: 161, 712
Istrăţescu, V. I. **1**: 485, 735
Its, A. **4**: 218, 695
Ivanov, V. I. **2A**: 350, 605
Ivriĭ, V. Ja. **4**: 604, 704
Iwamura, T. **4**: 467, 726
Iwaniec, T. **2A**: 128, 605
Izu, S. **3**: 337, 339, 712

Jackson, D. **1**: 81, 156, 163, 735; **4**: 267, 704
Jacobi, C. G. **1**: 567, 735; **2A**: 87, 304, 305, 315, 419, 450, 477, 497–499, 517, 533, 534, 550, 605, 606; **4**: 17, 241, 704, 705
Jaffe, A. **1**: 329, 731; **3**: 651, 654, 707
Jager, H. **3**: 125, 696
James, G. **4**: 443, 705
James, I. M. **1**: 367, 735
Janson, S. **3**: 653, 712
Janssen, A. J. E. M. **3**: 402, 403, 700, 712
Jarchow, H. **1**: 706, 735
Javrjan, V. A. **4**: 344, 705
Jensen, A. **4**: 534, 694
Jensen, J. L. **1**: 387, 666, 735; **2A**: 102, 450, 606
Jentzsch, R. **2A**: 239, 606; **3**: 654, 712
Jerison, M. **3**: 161, 712; **4**: 412, 700
Jessen, B. **3**: 48, 712
Jiang, B. **1**: 485, 720
Jitomirskaya, S. **1**: 702, 725; **3**: 294, 296, 514, 701, 712; **4**: 344, 695
Johansson, K. **1**: 630, 715
John, F. **1**: 606, 608, 736; **3**: 17, 534, 699, 712; **4**: 603, 694
Johnson, B. E. **4**: 388, 705
Johnson, W. B. **1**: 357, 736
Jonas, P. **4**: 353, 705
Jones, G. A. **2A**: 281, 333, 606
Jones, P. W. **4**: 389, 705
Jones, R. **3**: 291, 710
Jones, R. L. **3**: 84, 713
Jordan, C. **1**: 26, 50, 74, 152, 193, 269, 318, 736; **2A**: 87, 164, 606; **4**: 135, 603, 705
Jordan, P. **1**: 113, 117, 736
Jorgensen, P. **3**: 433, 696
Joricke, B. **3**: 333, 709
Joseph, A. **2A**: 306, 606
Jost, J. **1**: 453, 606, 736; **2A**: 21, 589, 606
Jost, R. **1**: 513, 736; **2A**: 195, 606
Joukowski, N. **2A**: 350, 606
Journé, J-L. **3**: 602, 700
Julia, G. **2A**: 573, 574, 606
Junek, H. **1**: 443, 736

Kérchy, L. **4**: 322, 722
Kac, I. **4**: 570, 705
Kac, M. **1**: 328, 736; **3**: 85, 713; **4**: 504, 605, 630, 705
Kadec, M. Ĭ **3**: 406, 713

Kadison, R. V. **1**: 92, 93, 736; **4**: 128, 314, 428, 700, 705, 706
Kahan, W. **4**: 135, 700
Kahane, J.-P. **1**: 153, 156, 736; **2A**: 58, 606; **3**: 437, 713; **4**: 392, 706
Kahn, J. **3**: 652, 654, 713
Kaiser, G. **3**: 433, 713
Kakeya, S. **3**: 684, 713
Kakutani, S. **1**: 89, 92, 211, 238, 298, 328, 444, 447, 486, 608, 726, 736, 737; **3**: 83, 734; **4**: 43, 706
Kalai, G. **3**: 652, 654, 713
Kalf, H. **4**: 569, 627, 696, 706
Kalikow, S. **3**: 79, 84, 97, 713
Kalisch, G. K. **4**: 129, 706
Kallenberg, O. **1**: 617, 737
Kalton, N. J. **1**: 357, 365, 444, 490, 713, 716, 737
Kamae, T. **3**: 145, 713
Kamran, N. **4**: 255, 700
Kaniuth, E. **4**: 357, 389, 706
Kannai, Y. **1**: 487, 737
Kannappan, Pl. **1**: 118, 737
Kantor, J-M. **1**: 249, 731
Kaplansky, I. **4**: 314, 406, 706
Karamata, J. **3**: 689, 713; **4**: 506, 706
Karatzas, I. **1**: 327, 737; **3**: 161, 713
Karunakaran, V. **2A**: 214, 606
Kasner, E. **2A**: 38, 606
Kato, T. **2A**: 131, 606; **3**: 337, 614, 713; **4**: 27, 70, 217, 343, 352, 353, 536, 537, 548, 600, 601, 627, 629, 706, 707
Katok, A. **3**: 84, 713
Katok, S. **2A**: 335, 606; **3**: 127, 713
Katz, N. H. **3**: 685, 696, 713
Katznelson, Y. **1**: 149, 153, 737; **3**: 145, 439, 713; **4**: 357, 367, 369, 707
Kaufman, R. **3**: 84, 713
Kawohl, B. **3**: 36, 713
Keane, M. **3**: 83, 123, 124, 145, 714
Kechris, A. S. **1**: 313, 737
Kečkić, J. D. **2A**: 214, 611
Keel, M. **3**: 683, 714
Keller, W. **3**: 433, 714
Kelley, J. L. **1**: 48, 98, 102, 106, 367, 464, 737; **4**: 428, 707
Kellogg, O. D. **3**: 177, 273, 274, 714
Kelton, N. J. **4**: 218, 707
Kelvin, Lord **2A**: 17, 315; **3**: 196, 273
Kemeny, J. G. **1**: 674, 737
Kemp, T. **3**: 653, 714
Kennard, E. H. **3**: 334, 714
Kennedy, P. B. **3**: 177, 253, 709
Kerber, A. **4**: 443, 705
Kesavan, S. **3**: 36, 714
Kesten, H. **3**: 145, 706
Khinchin, A. Ya. **1**: 227, 628, 645, 658, 737; **2A**: 304, 606; **3**: 83, 90, 123, 124, 714
Khoshnevisan, D. **1**: 617, 737
Khrushchev, S. **2A**: 305, 306, 607
Killing, W. **2A**: 404
Killip, R. **3**: 293, 700; **4**: 284, 707
Kim, S-h. **2A**: 333, 607
King, J. L. **3**: 99, 714
King, R. B. **2A**: 479, 607
Kingman, J. F. C. **3**: 145, 714
Kirchberger, P. **4**: 266, 707
Kirchoff, G. R. **1**: 607, 737
Kirillov, A., Jr. **4**: 217, 628, 707
Kirk, W. A. **1**: 485, 731, 737
Kirsch, W. **3**: 294, 700; **4**: 217, 218, 538, 694
Kiselev, A. **1**: 703, 737; **3**: 172, 698, 699, 714; **4**: 666, 707
Klauder, J. R. **3**: 385, 387, 401, 692, 693, 714
Klaus, M. **4**: 683, 707
Klee, V. L. **1**: 458, 737, 738
Klein, F. **2A**: 23, 266, 272, 282, 283, 292, 368, 404, 476, 480, 550, 568, 607
Knapp, A. W. **2A**: 477, 607; **3**: 613, 714; **4**: 443, 628, 707
Knaster, B. **1**: 50, 408, 738
Knopp, K. **2A**: 12, 420, 607; **3**: 125, 714
Knörrer, H. **2A**: 267, 595
Knuth, D. E. **2A**: 333, 602
Kobayashi, S. **2A**: 21, 607
Kober, H. **2A**: 350, 608
Koblitz, N. **2A**: 477, 550, 608
Koch, H. **2A**: 368, 608
Kodaira, K. **1**: 211, 737
Kodama, L. K. **4**: 490, 707
Koebe, P. **2A**: 238, 314, 367, 368, 608; **3**: 197, 316, 715
Koh, E. **3**: 172, 698
Kohn, J. J. **3**: 367, 715
Koksharov, R. **4**: 707
Kolk, J. A. C. **2A**: 398, 593; **4**: 628, 696

Kolmogorov, A. N. **1**: 61, 126, 153, 227, 298, 364, 627–629, 645, 658, 659, 674, 731, 738; **3**: 35, 65, 79, 162, 167, 463, 488, 715; **4**: 228, 388, 603, 699, 707
Kondrachov, V. I. **3**: 582, 715
Kondratiev, Yu. **1**: 313, 713
König, H. **1**: 608, 738; **4**: 187, 708
Koopman, B. O. **3**: 79–82, 125, 695, 715
Koosis, P. **3**: 439, 513, 534, 715; **4**: 389, 708
Koplienko, L. S. **4**: 353, 708
Koralov, L. B. **1**: 617, 738
Korevaar, J. **4**: 506, 708
Körner, T. W. **1**: 107, 149, 151, 355, 409, 738
Korovkin, P. P. **1**: 83, 738; **4**: 267, 708
Koshmanenko, Y. D. **4**: 666, 708
Kotani, S. **3**: 296, 715
Kotelnikov, V. A. **1**: 567, 739
Köthe, G. **1**: 443, 706, 711, 739; **2A**: 230, 608
Kovalevskaya, S. **2A**: 404
Kowa, S. T. **4**: 17, 708
Kozhan, R. **2A**: 456, 608
Kozitsky, Yu. **1**: 313, 713
Kra, I. **2A**: 267, 533, 589, 601; **3**: 316, 704
Kraaikamp, C. **2A**: 304, 605; **3**: 123, 700, 712
Krall, H. L. **4**: 254, 708
Krantz, S. G. **1**: 486, 700, 739; **2A**: 156, 323, 362, 468, 585, 602, 608; **3**: 47, 715
Krasnosel'skiĭ, M. **1**: 388, 739; **3**: 36, 715
Krasovsky, I. **4**: 218, 695
Kreicherbauer, T. **1**: 724
Krein, M. G. **1**: 92, 126, 433–435, 444, 464, 465, 713, 739; **4**: 134, 152, 153, 192, 218, 343, 353, 467, 468, 600, 601, 658, 687, 700, 708
Krein, S. G. **1**: 89, 92, 739; **3**: 556, 715
Krengel, U. **3**: 79, 145, 715
Kronecker, L. **1**: 13, 15, 487, 739; **2A**: 58; **3**: 98, 715
Krupnik, N. **3**: 603, 707; **4**: 187, 700
Kubrusly, C. S. **1**: 230, 739
Kufner, A. **3**: 336, 557, 715, 722
Kühnel, W. **2A**: 21, 608
Kuipers, L. **3**: 123, 715
Kumano-go, H. **3**: 367, 716
Kunen, K. **1**: 14, 739
Kunugi, K. **2A**: 152, 608
Kunze, R. A. **1**: 563, 739
Kurasov, P. **4**: 666, 687, 708
Kuratowski, K. **1**: 13, 48, 50, 60, 313, 407, 408, 738–740
Kuroda, S. T. **4**: 353, 534, 600, 666, 694, 707, 708
Kurzweil, J. **1**: 230, 740
Kuttler, K. **3**: 50, 716
Kuzmin, R. **3**: 124, 716
Kythe, P. K. **2A**: 350, 608

Łaba, L. **3**: 685, 716
Lacey, M. **3**: 172, 716
Lacroix, J. **3**: 294, 698
Laczkovich, M. **2A**: 305, 608
Lagrange, J.-L. **1**: 26, 150, 486, 740; **2A**: 57; **3**: 273; **4**: 17, 708
Laguerre, E. N. **2A**: 469, 474, 608
Lakey, J. **3**: 337, 339, 712
Lalesco, T. **4**: 163, 709
Lam, T. Y. **4**: 446, 709
Lambert, J. H. **2A**: 305, 608
Lamson, K. W. **1**: 364, 740
Landau, E. **1**: 9, 82, 162, 163, 740; **2A**: 12, 63, 118, 128, 238, 450, 468, 534, 577–579, 594, 596, 608, 609; **3**: 557, 716; **4**: 513, 605, 709
Landau, H. J. **3**: 337, 338, 716
Landkof, N. S. **1**: 453, 740; **2A**: 324, 609; **3**: 177, 276, 716
Lang, A. **1**: 660
Lang, S. **1**: 350, 351, 740; **2A**: 8, 12, 477, 609
Lapidus, M. L. **4**: 630, 709
Laplace, P.-S. **1**: 150, 606, 628, 653, 654, 740; **3**: 124, 249, 273, 716; **4**: 17, 709
Laptev, A. **3**: 340, 669, 670, 705, 711, 716
Larsen, R. **4**: 357, 709
Last, Y. **1**: 702, 703, 725, 737, 740; **2A**: 564, 592; **3**: 292, 514, 692, 701, 716; **4**: 666, 709
Laura, P. A. A. **2A**: 350, 617
Lavrentiev, M. A. **4**: 489, 709
Lawler, G. F. **1**: 327, 740
Lax, P. D. **1**: 185, 186, 225, 740; **3**: 367, 705; **4**: 217, 600, 709
Lay, S. R. **1**: 387, 740

Le Cam, L. **1**: 654, 657, 741
Lebesgue, H. **1**: 74, 82, 204, 229, 249, 257, 318, 408, 546, 701, 740, 741; **3**: 59, 231, 273, 716, 717; **4**: 256, 489, 709
Lebowitz, A. **2A**: 477, 615
Lee, S. **3**: 172, 682, 698, 717
Lee, T. D. **2A**: 239, 240, 609
Legendre, A.-M. **2A**: 304, 307, 419, 498, 517, 609; **3**: 249, 273, 717
Leibniz, G. W. **4**: 17, 709
Leibowitz, G. M. **2A**: 157, 609; **4**: 489, 709
Leighton, R. B. **1**: 588, 728
Leinfelder, H. **4**: 627, 709
Lemarié, P. G. **3**: 434, 717
Lemmermeyer, F. **2A**: 479, 609
Lenard, A. **3**: 344, 717
Lennes, N. J. **1**: 50, 741
Lenz, D. **1**: 411, 741
Leoni, G. **2A**: 165, 609; **3**: 583, 717
Leray, J. **1**: 487, 741; **2A**: 568
Lerch, M. **1**: 82, 741
Lesky, P. **4**: 255, 709
Lévy Véhel, J. **1**: 702
Levi, B. **1**: 249, 741
Levin, D. **3**: 669, 717
Levin, E. **3**: 292, 717
Levinson, N. **4**: 513, 569, 693, 709
Levitan, B. M. **4**: 419, 569, 710
Lévy, P. **1**: 327, 628, 654–659, 741, 742; **2A**: 470; **3**: 124, 162, 717; **4**: 388, 710
Lévy Véhel, J. **1**: 713
Lewis, J. L. **2A**: 574, 600; **3**: 218, 703, 717
Lewy, H. **1**: 608, 742
Li, B. R. **4**: 314, 710
Li, P. **3**: 669, 717
Li-Jost, X. **1**: 453, 736
Liao, M. **1**: 659, 742
Lidskii, V. B. **4**: 186, 710
Lie, S. **2A**: 404; **4**: 628, 710
Lieb, E. H. **1**: 249, 275, 394, 454, 563, 719, 742; **3**: 36, 275, 386, 563, 564, 653, 669, 691, 696, 698, 705, 711, 717, 718; **4**: 683, 710
Lifshitz, I. M. **4**: 353, 710
Liggett, T. M. **1**: 327, 742; **3**: 145, 161, 718
Light, W. **4**: 267, 693
Lin, F. **1**: 700, 742; **3**: 177, 708
Lindeberg, J. W. **1**: 654, 656, 742
Lindelöf, E. **1**: 52, 60, 74, 485, 742; **2A**: 12, 172, 173, 177, 214, 239, 609, 613
Lindenstrauss, J. **1**: 357, 444, 716, 736, 742; **4**: 43, 710
Lindley, D. **3**: 250, 718
Linial, N. **3**: 652, 654, 713
Lions, J.-L. **1**: 514; **2A**: 177, 609; **3**: 556, 718; **4**: 129, 600, 710
Liouville, J. **2A**: 87, 497, 610; **4**: 109, 721
Littlewood, J. E. **1**: 154, 249, 394, 582, 645, 733, 742; **2A**: 562; **3**: 36, 46, 52, 98, 213, 458, 464, 488, 557, 562, 564, 603, 709, 718; **4**: 367, 506, 702, 710
Livio, M. **2A**: 499, 610
Lizorkin, P. I. **3**: 583, 718
Loeb, P. A. **3**: 50, 705
Loève, M. **1**: 617, 655, 659, 742
Loewner, K. **4**: 606, 710
Löfström, J. **3**: 556, 583, 694
Lohwater, A. J. **2A**: 578, 610
Łojasiewicz, S. **1**: 608, 743
Lomonosov, V. I. **1**: 488, 743; **4**: 117, 118, 710
Loomann, H. **2A**: 68, 610
Loomis, L. H. **1**: 350, 565, 743; **2A**: 12, 17, 610; **3**: 513, 718; **4**: 468, 504, 710
López Safont, F. **3**: 556, 718
Lorch, E. R. **1**: 425, 743; **4**: 69, 313, 710
Lorentz, G. G. **1**: 83, 84, 490, 725, 743; **3**: 36, 37, 556, 718
Lorentz, H. A. **1**: 630; **4**: 603, 710
Loss, M. **1**: 275, 742; **3**: 36, 275, 564, 717
Loupias, G. **3**: 368, 708
Low, F. E. **3**: 402, 718
Löwig, H. **1**: 117, 425, 743
Lu, G. **3**: 682, 712
Lubinsky, D. S. **2A**: 564, 610; **3**: 292, 717, 718
Lucretius **1**: 326, 743
Luecking, D. H. **2A**: 229, 230, 233, 405, 610
Lumer, G. **4**: 489
Lusin, N. **1**: 226, 743
Luttinger, J. M. **1**: 394, 719; **3**: 563, 696

Lützen, J. **1**: 513, 743; **2A**: 87, 499, 610
Luxemburg, W. A. J. **1**: 269, 388, 743
Lyapunov, A. M. **1**: 628, 653, 655, 743
Lyons, R. **1**: 582, 743
Lyubarskii, Y. I. **3**: 401, 718

Mac Lane, S. **1**: 562, 744
Mackey, G. W. **1**: 443, 744; **4**: 314, 443, 711
Maclaurin, C. **2A**: 438, 610; **4**: 17, 711
MacRobert, T. M. **3**: 177, 718
Maggi, F. **3**: 654, 704
Makarov, N. G. **3**: 274, 718, 719; **4**: 344, 695
Malamud, M. **4**: 87, 699
Malgrange, B. **1**: 514, 607, 744
Maligranda, L. **1**: 372, 388, 744; **3**: 336, 557, 715
Mallat, S. **3**: 433, 434, 719
Malliavin, P. **1**: 230, 744
Mandelbrojt, S. **1**: 48
Mandelbrot, B. B. **1**: 679, 700, 744
Mandelkern, M. **1**: 62, 744
Manheim, J. H. **1**: 35, 744
Mansuy, R. **1**: 327, 744; **3**: 160, 719
Mantoiu, M. **4**: 666, 711
Marcellán, F. **4**: 255, 711
Marcinkiewicz, J. **3**: 48, 603, 712, 719
Marcon, D. **3**: 654, 712
Marcus, M. **3**: 336, 697
Markov, A. **1**: 81, 227, 238, 433, 486, 628, 653, 655, 674, 744; **2A**: 305; **4**: 241, 267, 711
Marks, R. J., II **1**: 568, 744
Markus, A. S. **1**: 394, 744
Markushevich, A. I. **2A**: 323, 502, 517, 574, 610
Marshall, A. W. **1**: 394, 744
Marshall, D. E. **3**: 274, 706; **4**: 389, 705
Martínez-Finkelshtein, A. **1**: 453, 745
Martin, A. **4**: 605, 711
Martin, G. **2A**: 128, 605
Martin, J. **1**: 630, 718
Martin, R. S. **3**: 276, 719
Martinelli, E. **2A**: 584, 610
Marty, F. **2A**: 239, 252, 610
Mascheroni, L. **2A**: 420
Maslov, V. P. **3**: 368, 719
Mason, J. C. **4**: 266, 711
Masters, W. **1**: 329, 721
Mather, J. N. **2A**: 324, 610
Matheson, A. L. **2A**: 188, 597; **3**: 489, 699
Mattila, P. **1**: 700, 745
Maurey, B. **1**: 514; **4**: 44, 100, 700, 701
Maz'ya, V. **2A**: 470, 610; **3**: 583, 719; **4**: 228, 603, 711
Mazo, R. M. **1**: 327, 745
Mazur, S. **1**: 357, 388, 458, 501, 716, 745; **4**: 357, 387, 711
Mazurkiewicz, S. **1**: 205, 745
McCarthy, J. **1**: 165, 745
McCutcheon, R. **3**: 79, 97, 713
McKean, H. P. **1**: 327, 537, 574, 727, 745; **2A**: 477, 610; **3**: 337, 702
McLauglin, K. T. R. **1**: 724
Medvedev, F. A. **1**: 155, 745
Meehan, M. **1**: 485, 713
Megginson, R. E. **1**: 444, 745; **4**: 100, 711
Mehrtens, H. **1**: 269, 745
Melas, A. D. **3**: 49, 719
Mellin, H. **1**: 548
Melnikov, A. **2A**: 306, 606
Menchoff, D. **2A**: 68, 610
Menger, K. **1**: 701, 745
Menshov, D. **3**: 172, 719
Meray, C. **1**: 9, 745
Mercer, J. **4**: 182, 711
Mergelyan, S. N. **2A**: 156, 610; **4**: 489, 711
Meyer, P.-A. **1**: 465, 723; **3**: 161, 177, 276, 701, 719
Meyer, Y. **3**: 433, 434, 614, 699, 719, 720
Meyer-Nieberg, P. **1**: 269, 745
Mhaskar, H. N. **1**: 83, 84, 745; **4**: 267, 711
Michaels, A. J. **4**: 119, 711
Michal, A. D. **1**: 365, 745
Michlin, S. G. **3**: 603, 720
Mikhailov, V. P. **1**: 606, 745
Mikosch, T. **1**: 659, 716
Mikusinski, J. G. **4**: 129, 711
Milgram, A. N. **4**: 600, 709
Miller, W. J. **4**: 443, 711
Milman, D. P. **1**: 444, 464, 739, 745
Milnor, J. **1**: 487, 745; **4**: 605, 712
Milson, R. **4**: 255, 700
Minda, D. **2A**: 378, 610
Minkowski, H. **1**: 371, 387, 464, 574, 745, 746; **2A**: 246, 404

Minlos, R. A. **1**: 565, 629, 746
Miranda, R. **2A**: 267, 589, 611
Mišik, L., Jr. **1**: 702, 746
Mitrea, M. **4**: 87, 602, 689, 699
Mitrinović, D. S. **1**: 388, 720; **2A**: 214, 611
Mittag-Leffler, G. **2A**: 400, 404, 409, 611
Miyake, T. **2A**: 550, 611
Mizohata, S. **1**: 608, 746
Mizuta, Y. **3**: 177, 720
Möbius, A. F. **2A**: 272, 282, 611
Mockenhaupt, G. **3**: 49, 683, 684, 720
Mohapatra, A. N. **4**: 353, 720
Molien, T. **4**: 446, 712
Moll, V. **2A**: 477, 610
Mollerup, J. **2A**: 420, 594
Monge, G. **2A**: 272
Montanaro, A. **3**: 654, 720
Montel, P. **1**: 74; **2A**: 68, 238, 611
Montesinos, V. **1**: 357, 444, 728
Montgomery, H. L. **3**: 123, 129, 720
Montiel, S. **3**: 17, 720
Moonen, M. S. **4**: 135, 712
Moore, C. D. **2A**: 304, 611
Moore, E. H. **1**: 98, 746
Moore, G. H. **1**: 48, 746
Moore, R. L. **1**: 106, 746
Moral, L. **4**: 284, 692
Moran, W. **1**: 582, 720
Morawetz, C. S. **3**: 682, 720
Mordell, L. J. **2A**: 58, 64, 518, 611
Morera, G. **2A**: 87, 611
Morgan, F. **1**: 700, 746
Morgan, G. W. **3**: 337, 720
Morgan, J. **3**: 654, 720
Morlet, J. **3**: 386, 387, 708
Morrey, Ch. B., Jr. **3**: 581, 720
Morris, S. A. **2A**: 68, 602
Morse, A. P. **3**: 50, 720
Morse, M. **3**: 83
Mörters, P. **1**: 327, 328, 746
Morton, P. **4**: 370, 692
Mosak, R. D. **4**: 357, 712
Moschovakis, Y. N. **1**: 14, 746
Moser, J. **3**: 653, 720
Moslehian, M. S. **4**: 44, 712
Moyal, J. E. **3**: 370, 386, 720
Mueller, P. **3**: 407, 433, 733
Muir, T. **4**: 17, 712
Muirhead, R. F. **1**: 394, 746; **3**: 36, 720
Mumford, D. **2A**: 281, 335, 611
Munkres, J. R. **1**: 61, 746
Müntz, C. **2A**: 456, 458, 611
Murnaghan, F. D. **4**: 82, 726
Murphy, G. **4**: 314, 712
Murray, F. J. **1**: 182, 425, 746; **4**: 43, 712
Muscalu, C. **3**: 682, 721
Mushtari, D. H. **1**: 313, 746
Muskhelishvili, N. I. **3**: 603, 721
Mycielski, J. **1**: 12, 746; **3**: 335, 694
Myers, D. L. **1**: 230, 762
Myland, J. **2A**: 214, 612

Naboko, S. **4**: 87, 699
Nachbin, L. **1**: 350, 514, 746
Nadkarni, M. G. **3**: 79, 721
Nagata, J. **1**: 61, 746
Nagumo, M. **4**: 56, 69, 357, 712
Nahin, P. J. **2A**: 17, 611
Naimark, M. A. **4**: 357, 399, 405, 406, 428, 447, 504, 569, 659, 699, 712
Najmi, A-H. **3**: 433, 721
Nakano, H. **4**: 313, 712
Napier, T. **2A**: 589, 611
Narasimhan, R. **2A**: 17, 68, 585, 589, 612; **4**: 389, 712
Narcowich, F. **3**: 433, 695
Narici, L. **1**: 443, 706, 746
Nash, J. **3**: 582, 653, 721
Nason, G. P. **3**: 433, 721
Naumann, J. **3**: 581, 721
Nazarov, F. L. **3**: 337, 721
Nehari, Z. **2A**: 350, 362, 612; **3**: 535, 721
Neidhardt, H. **4**: 353, 630, 697, 712
Nekrasov, P. A. **1**: 674, 746
Nelson, E. **1**: 327, 329, 747; **3**: 197, 651, 652, 721; **4**: 323, 600, 630, 712
Neretin, Y. A. **1**: 538, 747
Netuka, I. **3**: 197, 721
Neuenschwander, E. **2A**: 87, 128, 612
Neumann, C. G. **2A**: 131, 612; **3**: 273, 275, 721; **4**: 56, 118, 712
Nevai, P. **4**: 282, 689
Nevanlinna, F. **3**: 444, 457, 721
Nevanlinna, R. **1**: 60, 433, 747; **2A**: 451, 612; **3**: 197, 444, 457, 513, 721; **4**: 658, 713
Neveu, J. **3**: 161, 722
Neville, E. H. **2A**: 477, 612
Newcomb, S. **3**: 100, 722

Newman, D. J. **4**: 43, 390, 713
Newman, F. W. **2A**: 419, 612
Newton, I. **1**: 453, 486, 747; **2A**: 518
Niederreiter, H. **3**: 123, 715
Nievergelt, Y. **2A**: 68, 612; **3**: 433, 722; **4**: 389, 712
Nigrini, M. J. **3**: 100, 722
Nikishin, E. M. **4**: 231, 713
Nikodym, O. **1**: 257, 364, 747; **3**: 581, 722
Nikolsky, S. M. **4**: 603, 713
Nirenberg, L. **3**: 352, 367, 534, 582, 712, 715, 722
Nittka, R. **4**: 604, 688
Noether, E. **2A**: 26, 612; **3**: 543
Noether, F. **4**: 216, 713
Nomizu, K. **2A**: 21, 607
Nonnenmacher, S. **4**: 605, 713
Norris, J. R. **1**: 674, 747
Novinger, W. P. **2A**: 150, 323, 592
Nyquist, H. **1**: 567, 747

O'Neil, R. **3**: 557, 722
O'Regan, D. **1**: 485, 713
Oberhettinger, F. **1**: 630, 747
Odake, S. **4**: 255, 713
Oguntuase, J. A. **3**: 557, 722
Ogura, K. **1**: 569, 747; **2A**: 221, 612
Ohtsuka, M. **2A**: 323, 612
Oldham, K. **2A**: 214, 612
Olds, C. D. **2A**: 304, 612
Olkiewicz, R. **3**: 653, 722
Olkin, I. **1**: 394, 744
Opic, B. **3**: 336, 557, 722
Orlicz, W. **1**: 388, 501, 717, 745, 747; **3**: 36, 722
Ornstein, D. S. **3**: 65, 83, 86, 97, 698, 722
Ortega-Cerdà, J. **3**: 406, 722
Ortner, N. **1**: 608, 747
Osborne, M. S. **1**: 443, 747
Oscledec, V. I. **3**: 145, 722
Osgood, B. **2A**: 117, 612
Osgood, W. F. **1**: 407, 747; **2A**: 79, 87, 128, 238, 314, 323, 612
Östlund, S. **2A**: 333, 607
Ostrowski, A. **2A**: 315, 613
Otto, F. **3**: 654, 722
Outerelo, E. **1**: 487, 747
Oxtoby, J. C. **1**: 408, 747

Pai, D. V. **1**: 83, 84, 745; **4**: 267, 711
Painlevé, P. **1**: 74; **2A**: 323, 574, 613
Pajot, H. **2A**: 128, 378, 613
Pál, J. **1**: 84, 747
Palais, R. S. **3**: 367, 722
Paley, R. E. A. C. **1**: 328, 747; **2A**: 58, 135, 562, 613; **3**: 406, 464, 603, 718, 722, 723; **4**: 367, 467, 713
Palmer, T. W. **4**: 357, 713
Pareto, V. **1**: 658, 747
Parker, I. B. **4**: 266, 697
Parks, H. R. **1**: 486, 700, 739
Parry, W. **3**: 79, 123, 723
Parseval, M.-A. **1**: 150, 607, 748
Parzen, E. **1**: 126, 748
Pascal, B. **1**: 628
Pastur, L. A. **3**: 294, 723
Patashnik, O. **2A**: 333, 602
Patodi, V. K. **4**: 217, 689
Paul, T. **3**: 386, 708
Pauli, W. **4**: 27, 713
Peano, G. **1**: 9, 26, 748; **4**: 603, 713
Pearcy, C. **1**: 488, 748; **4**: 119, 713
Pearson, D. B. **4**: 666, 713
Pečarić, J. E. **1**: 387, 748
Pedersen, G. K. **1**: 350, 748; **4**: 314, 713
Peetre, J. **3**: 556, 718, 723
Peirce, C. S. **1**: 9, 748
Peller, V. V. **3**: 536, 723; **4**: 218, 353, 713
Percival, D. B. **3**: 433, 723
Perelman, G. **3**: 654, 723
Perelomov, A. M. **3**: 386, 401, 723
Peres, Y. **1**: 327, 328, 746
Pérez Carreras, P. **1**: 443, 748
Perron, O. **1**: 230, 675, 748; **2A**: 583; **3**: 212, 231, 273, 723
Persson, L. E. **3**: 336, 557, 715, 722
Pesic, P. **2A**: 499, 613
Peter, F. **4**: 447, 713
Peter, W. **4**: 604, 688
Petersen, K. **3**: 79, 83, 145, 714, 723
Petersen, P. **2A**: 21, 613
Petronilho, J. **4**: 255, 711
Pettis, B. J. **1**: 275, 341, 444, 726, 748
Peyrière, J. **1**: 582, 748
Phelps, R. R. **1**: 387, 465, 468, 749; **4**: 489, 714
Philipp, W. **3**: 123, 723
Phillips, J. **3**: 387, 723
Phillips, R. S. **1**: 443, 749; **3**: 83, 723; **4**: 43, 45, 714

Phong, D. H. **3**: 336, 704; **4**: 228, 697
Phragmén, E. **2A**: 172, 173, 613
Picard, É. **1**: 74, 82, 161, 485, 749; **2A**: 12, 409, 573, 613; **3**: 197, 723; **4**: 83, 134, 714
Pichorides, S. K. **3**: 488, 723
Pick, G. **3**: 513, 723
Pier, J.-P. **1**: 486, 749
Pietsch, A. **1**: 363, 443, 447, 749
Pinkus, A. **1**: 81, 83, 156, 749
Pinsky, M. A. **3**: 433, 723
Pisier, G. **1**: 511, 513, 514, 759; **4**: 100, 187, 714
Pitt, H. R. **4**: 504, 714
Plamenevskii, B. A. **3**: 367, 723
Plancherel, M. **1**: 151, 546, 749
Plato **2A**: 1, 613
Plemelj, J. **1**: 512, 749; **3**: 489, 724; **4**: 172, 714
Plesner, A. I. **3**: 463, 724; **4**: 314, 714
Poincaré, H. **1**: 37, 47, 355, 486, 656, 705, 749; **2A**: 23, 25, 37, 272, 282, 292, 314, 367, 368, 469, 568, 584, 613, 614; **3**: 80, 85, 212, 231, 273, 275, 316, 581, 724; **4**: 192, 355, 628, 714
Poisson, S. D. **1**: 567, 606, 607, 644, 666, 749, 750; **2A**: 180, 419, 614; **3**: 197, 273, 724
Polishchuk, A. **2A**: 534, 614
Pollak, H. O. **3**: 337, 338, 716, 729
Pollicott, M. P. **3**: 79, 724
Poltoratski, A. **3**: 64, 514, 724
Pólya, G. **1**: 153, 394, 653, 654, 657, 733, 750; **2A**: 214, 387, 468, 614; **3**: 36, 488, 557, 564, 709; **4**: 282, 714
Pommerenke, C. **2A**: 323, 324, 378, 578, 610, 614
Pompeiu, D. **2A**: 78, 194, 614
Ponce, G. **4**: 534, 694
Poncelet, J.-V. **2A**: 272
Pontryagin, L. S. **4**: 367, 467, 714
Port, S. C. **1**: 608, 750; **3**: 177, 724
Porter, M. B. **2A**: 58, 238, 614
Possel, R. **1**: 48
Post, K. A. **3**: 99, 701
Potapov, V. P. **2A**: 456, 614
Povzner, A. Ya. **1**: 565, 750; **4**: 467, 714
Pratelli, A. **3**: 654, 704
Pressley, A. **2A**: 21, 614
Priestley, H. A. **1**: 230, 750
Pringsheim, A. **2A**: 63, 68, 405, 583, 614
Privalov, I. I. **3**: 489, 724
Prokhorov, A. **1**: 629
Prokhorov, Yu. V. **1**: 313, 750
Proschan, F. **1**: 387, 748
Prössdorf, S. **3**: 603, 720
Prüfer, H. **2A**: 305
Prym, F. E. **2A**: 430, 615
Ptolemy, C. **2A**: 272
Puiseux, V. A. **2A**: 113, 164, 266, 615
Pushnitski, A. **4**: 344, 352, 353, 699
Putnam, C. R. **4**: 666, 715

Qi, F. **2A**: 447, 622
Quéfflec, M. **3**: 97, 724
Quesne, C. **4**: 255, 715

Rabinovich, V. S. **4**: 666, 715
Rademacher, H. **1**: 574, 750; **2A**: 222, 304, 333, 615; **3**: 409, 725
Radjavi, H. **4**: 119, 715
Radó, T. **1**: 153; **2A**: 266, 315, 356, 615
Radon, J. **1**: 229, 257, 548, 750
Raghunathan, M. S. **3**: 145, 725
Raikov, D. **1**: 564, 565, 666, 750; **4**: 357, 387, 399, 406, 447, 467–469, 489, 504, 699, 715
Rajchman, A. **1**: 582, 751
Rakhmanov, E. A. **3**: 292, 725
Ramachandran, M. **2A**: 589, 611
Ramey, W. **3**: 177, 692
Range, R. M. **2A**: 584, 585, 615
Ranicki, A. **2A**: 164, 615
Ransford, T. **1**: 453, 751; **2A**: 324, 615; **3**: 177, 274, 725
Rao, M. M. **1**: 230, 751; **3**: 161, 725
Rauch, H. E. **2A**: 477, 534, 615
Rauzy, G. **3**: 123, 725
Ravetz, J. R. **1**: 150, 751
Rayleigh, Lord **4**: 27, 109, 603, 715
Read, C. J. **1**: 488, 751
Reed, M. **1**: 538, 566, 675, 751; **3**: 654, 683, 725; **4**: 27, 353, 568, 569, 600, 629, 667, 715
Regev, O. **3**: 654, 693
Reich, S. **1**: 485, 751
Reid, C. **2A**: 369, 615; **4**: 41, 43, 715
Reiner, I. **4**: 443, 694
Reingold, N. **3**: 83, 725
Reinov, O. **4**: 187, 715
Reinsch, C. **4**: 135, 700

Reiter, H. **4**: 468, 715
Rellich, F. **1**: 122, 751; **3**: 582, 725; **4**: 27, 228, 536, 548, 603, 715, 716
Remling, C. **3**: 293, 725
Remmert, R. **2A**: 58, 87, 159, 227, 315, 405, 420, 579, 615
Renardy, M. **1**: 606, 751
Rentschler, R. **2A**: 306, 606
Rényi, A. **1**: 646, 727
Resnick, S. **1**: 659, 716
Retherford, J. R. **4**: 134, 716
Revuz, D. **1**: 327, 674, 751
Rezende, J. **3**: 669, 695
Ribarič, M. **4**: 200, 716
Rice, A. **2A**: 477, 615
Richards, I. **1**: 518, 751
Rickart, C. E. **4**: 357, 405, 716
Rickman, S. **2A**: 574, 615; **3**: 218, 725
Riemann, G. F. B. **1**: 142, 193, 228, 546, 607, 751; **2A**: 21, 37, 38, 50, 117, 127, 265, 314–316, 323, 534, 568, 589, 615, 616; **3**: 197, 273, 274, 725
Riesz, F. **1**: 6, 47, 50, 51, 74, 92, 117, 122, 124, 125, 150, 153, 193, 226, 229, 238, 249, 250, 269, 275, 363, 364, 371, 372, 447, 565, 581, 751, 752; **2A**: 315; **3**: 46, 51, 59, 212, 213, 273, 274, 434, 444, 457, 463, 513, 562, 725, 726; **4**: 43, 69, 83, 100, 118, 299, 321, 716
Riesz, M. **1**: 153, 238, 424, 433, 563, 752; **2A**: 177, 242, 404, 616; **3**: 274, 276, 457, 487, 488, 497, 603, 726; **4**: 228, 716
Ringrose, J. R. **1**: 444, 752; **4**: 127, 128, 314, 705, 716
Rinow, W. **2A**: 20, 605
Ritz, W. **4**: 109, 716
Rivlin, T. J. **4**: 266, 267, 716
Robert, D. **4**: 228, 604, 716
Roberts, A. W. **1**: 387, 752
Robertson, A. P. **1**: 706, 752
Robertson, H. P. **3**: 82, 334, 726
Robertson, W. **1**: 706, 752
Robin, G. **1**: 453, 752; **3**: 274, 726
Robinson, A. **1**: 487, 717
Robinson, D. W. **4**: 314, 601, 605, 691, 716
Robinson, R. M. **2A**: 577
Rockafellar, R. T. **1**: 387, 752
Rockett, A. M. **3**: 123, 726
Röckner, M. **1**: 313, 713
Röding, E. **2A**: 378, 616
Roepstorff, G. **1**: 327, 752
Rogava, D. L. **4**: 630, 716
Rogers, C. A. **1**: 487, 700, 702, 753; **3**: 564, 726
Rogers, L. J. **1**: 372, 753
Rogers, R. C. **1**: 606, 751
Rogosinski, W. W. **2A**: 182, 616
Rohlin, V. A. **4**: 314, 714
Rollnik, H. **4**: 683, 717
Romberg, J. **3**: 339, 698
Ros, A. **2A**: 578, 616; **3**: 17, 720
Rosay, J.-P. **1**: 607, 753
Rosen, J. **3**: 653, 726
Rosenblatt, J. M. **3**: 84, 713
Rosenbloom, P. C. **1**: 608, 753
Rosenblum, M. **4**: 353, 717
Rosenthal, A. **4**: 489, 702
Rosenthal, P. **4**: 119, 715
Ross, K. A. **4**: 443, 468, 504, 505, 703
Ross, W. T. **2A**: 188, 597; **3**: 489, 699
Rossi, H. **2A**: 585, 603
Rota, G-C. **3**: 161, 406, 695, 726
Roth, A. **4**: 490, 717
Rothaus, O. S. **3**: 652, 726
Rothe, H. A. **2A**: 534, 537, 616
Rouché, E. **2A**: 100, 616
Routh, E. **4**: 254, 717
Roy, R. **2A**: 419, 421, 534, 535, 592; **4**: 231, 254, 688
Royer, G. **3**: 650, 726
Rozenbljum, G. V. **3**: 669, 726
Rubel, L. A. **2A**: 161, 229, 230, 233, 405, 610, 616
Rubin, J. E. **1**: 13, 735
Ruch, D.-K. **3**: 433, 726
Rudin, W. **1**: 565, 753; **2A**: 194, 195, 585, 616; **3**: 439, 472, 701, 726; **4**: 43, 357, 369, 468, 504, 505, 717
Ruelle, D. **3**: 145, 726; **4**: 321, 717
Ruiz, J. M. **1**: 487, 747
Rumin, M. **3**: 670, 727
Runde, V. **1**: 486, 753
Runge, C. **2A**: 156, 409, 616
Runst, T. **3**: 583, 727
Ruston, A. F. **4**: 183, 717
Rutickiĭ, Ya. **1**: 388, 739; **3**: 36, 715
Ryll-Nardzewski, C. **3**: 124, 727

Saalschütz, L. **2A**: 430, 616
Sadosky, C. **3**: 603, 614, 727
Saff, E. B. **1**: 453, 753; **4**: 695
Sagan, H. **1**: 204, 753
Sagher, Y. **3**: 534, 700
Saint-Raymond, X. **3**: 367, 727
Saitoh, S. **1**: 126, 753
Sakai, S. **4**: 314, 429, 717
Saks, S. **1**: 238, 753; **2A**: 149, 157, 238, 574, 616; **3**: 64, 727
Salminen, P. **1**: 327, 719
Saloff-Coste, L. **3**: 653, 702
Sands, M. **1**: 588, 728
Sarason, D. **3**: 534, 727; **4**: 128, 490, 658, 687, 717
Sargsjan, I. S. **4**: 569, 710
Sasaki, R. **4**: 255, 713
Sasvári, Z. **2A**: 447, 617
Sato, M. **3**: 350, 727
Schaefer, H. H. **1**: 269, 706, 753
Schaeffer, A. C. **3**: 401, 403, 702
Schatten, R. **1**: 182, 753; **4**: 143, 152, 717
Schatz, J. **4**: 428, 717
Schauder, J. **1**: 408, 487, 741, 753; **4**: 43, 100, 118, 717
Schechter, E. **1**: 12, 753
Schechter, M. **4**: 717
Scheffé, H. **1**: 249, 753
Scheidemann, V. **2A**: 584, 585, 617
Schiefermayr, K. **4**: 267, 718
Schiff, J. L. **2A**: 573, 574, 578, 617
Schiffer, M. **1**: 126, 717
Schinzinger, R. **2A**: 350, 617
Schlag, W. **3**: 682, 683, 707, 712, 721
Schlömilch, O. **2A**: 419, 617
Schmüdgen, K. **4**: 602, 718
Schmeisser, G. **1**: 569, 720
Schmidt, E. **1**: 117, 122, 132, 134, 753; **4**: 83, 99, 100, 108, 134, 299, 718
Schmincke, U.-W. **4**: 627, 718
Schneider, R. **1**: 167, 732
Schönflies, A. M. **1**: 15, 50, 74, 117, 754; **2A**: 404
Schottky, F. **2A**: 394, 404, 577, 578, 617
Schoutens, W. **1**: 659, 754
Schrader, R. **4**: 627, 628, 703
Schrödinger, E. **1**: 607, 754; **2A**: 266; **3**: 334, 727; **4**: 27, 718
Schulze, B.-W. **3**: 367, 703
Schur, I. **1**: 175, 350, 394, 444, 754; **2A**: 239, 305, 617; **3**: 488, 727; **4**: 163, 208, 446, 718
Schwartz, J. T. **1**: 487, 726; **3**: 86, 702; **4**: 186, 192, 568, 569, 600, 696, 718
Schwartz, L. **1**: 126, 512, 513, 565, 711, 712, 725, 754; **2A**: 562, 617
Schwarz, H. A. **1**: 112, 117, 754; **2A**: 117, 180, 181, 194, 283, 314, 351, 404, 568, 617; **3**: 273
Schwerdtfeger, H. **2A**: 281, 617
Schwinger, J. **4**: 682, 718
Sebestyén, Z. **4**: 601, 702
Seco, L. A. **4**: 603, 703
Seebach, J. A., Jr. **1**: 408, 756
Seeger, A. **3**: 684, 720
Seeley, R. T. **3**: 367, 727
Segal, I. E. **1**: 538, 754; **3**: 385, 652, 683, 727; **4**: 299, 428, 429, 447, 504, 718
Segal, S. L. **2A**: 468, 502, 617
Seidel, W. **2A**: 161, 617
Seifert, H. **1**: 106, 754
Seiler, E. **4**: 161, 172, 192, 718, 719
Seiler, R. **4**: 217, 689
Seip, K. **3**: 401, 406, 722, 728
Seiringer, R. **3**: 669, 705
Selberg, A. **2A**: 419, 617
Selçuk, F. **3**: 433, 706
Semenov, E. M. **3**: 556, 715
Seneta, E. **1**: 674, 755
Series, C. **2A**: 281, 333, 335, 611, 617; **3**: 126, 696, 728
Serre, J.-P. **2A**: 550, 617; **4**: 443, 719
Serrin, J. **2A**: 87, 618
Severini, C. **1**: 249, 755
Shafer, G. **3**: 160, 694
Shakarchi, R. **1**: 149, 409, 756; **3**: 487, 682, 730
Shannon, C. E. **1**: 567, 755; **3**: 334, 728
Shapiro, H. S. **4**: 719
Shaposhnikova, T. **2A**: 470, 610
Sharpley, R. **3**: 534, 556, 583, 694
Shelley, P. B. **3**: 319, 728
Shen, A. **1**: 14, 755; **3**: 160, 694
Shenitzer, A. **4**: 603, 719
Shields, A. L. **1**: 488, 748; **4**: 119, 713
Shilov, G. E. **1**: 513, 548, 730; **4**: 69, 357, 387, 389, 406, 467, 489, 504, 699, 719
Shiryayv, A. N. **1**: 617, 755

Shmulyan, V. **1**: 447, 755
Shohat, J. A. **1**: 434, 755
Shoikhet, D. **1**: 485, 751
Shokrollahi, M. A. **2A**: 112, 595
Shreve, S. E. **1**: 327, 737; **3**: 161, 713
Shterenberg, R. **4**: 602, 689
Shubin, M. A. **3**: 367, 368, 371, 728
Shurman, J. **2A**: 550, 598
Shvartsman, P. **3**: 534, 700
Sickel, W. **3**: 583, 727
Siegel, C. L. **1**: 574, 755; **2A**: 477, 618
Siegmund-Schultze, R. **1**: 562, 755
Sierpinski, W. **1**: 60, 740; **3**: 97, 98, 728; **4**: 605, 719
Šikić, H. **1**: 519, 734
Silbermann, B. **4**: 218, 691
Silva, C. E. **3**: 79, 728
Silverman, J. H. **2A**: 518, 550, 618
Silverstein, M. L. **3**: 162, 697
Simader, C. G. **4**: 627, 709
Simon, B. **1**: 135, 290, 327–329, 387, 388, 394, 411, 434–436, 443, 453, 454, 464, 465, 537, 538, 564, 566, 582, 617, 675, 702, 703, 721, 725, 734, 737, 742, 751, 755; **2A**: 58, 59, 239, 241, 242, 286, 289, 306, 335, 350, 535, 564, 592, 595, 618; **3**: 36, 127, 146, 197, 250, 291–297, 336, 340, 386, 387, 472, 514, 563, 650–654, 669, 670, 683, 689, 692, 699–701, 704, 705, 711, 712, 716, 724, 725, 728, 729; **4**: 27, 134, 152, 161, 172, 187, 192, 217, 218, 228, 231, 267, 282, 284, 285, 323, 344, 352, 353, 443, 446, 509, 534, 538, 568–570, 600–604, 606, 608, 609, 626–629, 658, 659, 666, 667, 682, 683, 685, 687–689, 691, 693–695, 699, 703, 704, 707, 709, 715, 718–720
Simon, L. **1**: 700, 755
Sims, B. **1**: 737
Sinai, Ya. G. **1**: 617, 629, 738; **3**: 79, 729
Singer, I. M. **4**: 100, 128, 217, 489, 689, 706, 720
Singerman, D. **2A**: 281, 333, 606
Sinha, K. B. **4**: 218, 353, 354, 688, 693, 720
Sitaram, A. **3**: 333, 338, 342, 704
Sjöstrand, J. **1**: 539, 756; **4**: 605, 713
Skolem, T. **1**: 13, 756
Slepian, D. **3**: 338, 729
Sleshinskii, I. V. **1**: 655, 756
Smart, D. R. **1**: 485, 756
Smirnov, V. I. **3**: 470, 729
Smirnov, Yu. **1**: 61, 756
Smith, C. **3**: 250, 729
Smith, H. J. S. **1**: 201, 756
Smith, H. L. **1**: 98, 746
Smith, K. T. **1**: 487, 715; **3**: 276, 681, 692; **4**: 627, 688
Smith, P. A. **3**: 80, 695
Smithies, F. **2A**: 39, 618; **4**: 172, 192, 721
Šmulian, V. **1**: 465, 739
Snell, J. L. **1**: 674, 737
Sobczyk, A. **1**: 425, 718
Sobolev, S. L. **1**: 512, 607, 756; **3**: 562, 582, 729, 730
Sodin, M. **2A**: 578
Sodin, M. L. **3**: 218, 703; **4**: 256, 721
Sogge, C. D. **3**: 564, 684, 720, 730
Sokhotskii, Yu. V. **1**: 512, 756; **2A**: 128, 618
Solomyak, M. **3**: 340, 669, 716, 717; **4**: 160, 353, 690, 721
Solovay, R. M. **1**: 211, 756
Sommerfeld, A. **4**: 603, 721
Song, R. **3**: 162, 701
Soper, H. E. **1**: 666, 756
Sørensen, H. K. **2A**: 499, 618
Sorokin, V. N. **4**: 231, 713
Soukhomlinoff, G. **1**: 425, 756
Souslin, M. Y. **1**: 227
Spanier, J. **2A**: 214, 612
Spanne, S. **3**: 489, 730
Spencer, J. H. **1**: 617, 714
Spencer, T. **1**: 608, 730
Spitzer, F. **3**: 164, 730
Spivak, M. **2A**: 17, 21, 618
Spivey, M. Z. **2A**: 444, 618
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Srinivasa Rao, K. N. **2A**: 214, 618
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Barry Simon is currently an IBM Professor of Mathematics and Theoretical Physics at the California Institute of Technology. He graduated from Princeton University with his Ph.D. in physics. In 2012 Simon won the International Association of Mathematical Physics' Poincaré Prize for outstanding contributions to mathematical physics. He has authored more than 400 publications on mathematics and physics.

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